LECTURE 9

EXTENSION PROBLEMS

IN

SOBOLEV SPACES :

SKETCH OF PROOFS

This talk sketches the proofs of some of the results stated last time.

WE WORK IN $L^{m,p}(\mathbb{R}^n)$ (HOMOGENEOUS SOBOLEV SPACE).

ANALOGOUS RESULTS in $W^{m,p}$ (INHOMOGENEOUS SPACE) FOLLOW EASILY.

WE SUPPOSE THROUGOUT

THAT $p > n$.

Constants $c, C, C' \ldots$

WILL AWAYS DEPEND ONLY ON

m, n, p (in $L^{m,p}(\mathbb{R}^n)$)

UNLESS WE SAY OTHERWISE.

These symbols may denote

different constants in

different occurrences.

A BRIEF REVIEW of

BASIC DEFINITIONS:

LET $E \subset \mathbb{R}^n$ (FINITE).

$L^{m,p}(E) = \{$ALL FNS. $f: E \to \mathbb{R}\}$

with SEMINORM

$$\|f\|_{L^{m,p}(E)} = \inf\{\|F\|_{L^{m,p}(\mathbb{R}^n)} : F = f \text{ on } E\}$$

A LINEAR EXTENSION OPERATOR

for E is a linear map

$$T: L^{m,p}(E) \to L^{m,p}(\mathbb{R}^n)$$

such that

$$Tf = f \text{ on } E$$

for all $f \in L^{m,p}(E)$.

A <u>SET OF ASSISTS</u> for E

is a set $\Omega = \{\omega_1, \cdots, \omega_S\}$

of linear functionals on $L^{m,p}(E)$,

with each ω_s given by

$$\omega_s(f) = \sum_{y \in E} \mu_s(y) f(y)$$

for coeff's $\mu_s(y)$ independent of f,

s.t. the <u>total</u> number of

nonzero $\mu_s(y)$ $(1 \le s \le S,\ y \in E)$

is at most CN $\quad (N = \#(E))$.

Let $\Omega = \{\omega_1, \cdots, \omega_S\}$

be a set of assists.

A linear functional

$\xi : L^{m,p}(E) \to \mathbb{R}$

has Ω-<u>ASSISTED BDD DEPTH</u>

("ABD") if it can be

written in the form:

$$\boxed{\xi(f) = \sum_{y \in E} \mu(y) f(y) + \sum_{s=1}^{\bar{S}} \lambda_s \omega_s(f)}$$

with coeff's $\mu(y)$ $(y \in E)$

and λ_s $(s = 1, \dots, \bar{S})$

independent of f,

& s.t.

At most C of the $\mu(y)$

and

At most C of the λ_s

are NONZERO.

Let $\Omega = \{\omega_1, \dots, \omega_s\}$ be

a set of assists, and

let

$$T : L^{m,p}(E) \longrightarrow L^{m,p}(\mathbb{R}^n)$$

be a linear map.

Then T is of

<u>Ω-ASSISTED BDD DEPTH ("ABD")</u>

if ....

FOR EACH $x \in \mathbb{R}^n$

WE CAN WRITE

$$\boxed{Tf(x) = \sum_{y \in E} K(x,y) f(y) + \sum_{s=1}^{S} \beta_s(x) \omega_s(f)}$$

with $K(x,y)$, $\beta_s(x)$ INDEP. OF f,

and with

AT MOST C NONZERO $K(x,y)$

and

AT MOST C NONZERO $\beta_s(x)$

for each fixed x.

Today, $J_x(F)$ DENOTES the Taylor poly. of DEGREE $(m-1)$ (NOT m) at F at x.

That's NATURAL, because $L^{m,p}(\mathbb{R}^n) \subset C^{m-1}_{loc}(\mathbb{R}^n)$ for $p > n$.

$\mathcal{P}$ = VECTOR SPACE OF ALL $(m-1)^{\text{st}}$ DEGREE POLYS

TODAY'S THM.:

LET $p > n$, $E \subset \mathbb{R}^n$, $N = \#(E) < \infty$.

THEN THERE EXIST

$$\boxed{\text{A SET OF ASSISTS } \Omega = \{\omega_1, \dots, \omega_S\}}$$

$$\boxed{\begin{array}{l}\text{LINEAR FUNCTIONALS} \\ \xi_1, \dots, \xi_L : L^{m,p}(E) \to \mathbb{R}\end{array}}$$

$$\boxed{\begin{array}{l}\text{A LINEAR EXTENSION OPERATOR} \\ T : L^{m,p}(E) \to L^{m,p}(\mathbb{R}^n)\end{array}}$$

WITH THE FOLLOWING PROPERTIES:

PROPERTIES OF T

- $\|Tf\|_{L^{m,p}(\mathbb{R}^n)} \leq C\|f\|_{L^{m,p}(E)}$

  for all $f \in L^{m,p}(E)$

- T is of Ω- ASSISTED BDD DEPTH.

COMPUTATION OF THE NORM

For $f \in L^{m,p}(E)$, we have

$$c |||f||| \leq \|f\|_{L^{m,p}(E)} \leq C |||f|||$$

where

$$|||f||| = \left(\sum_{\ell=1}^{L} |\xi_\ell(f)|^p \right)^{1/p}.$$

MOREOVER,

$$\boxed{\text{Each } \xi_\ell \ (\ell = 1, \dots, L) \text{ has } \Omega\text{- ASSISTED BDD. DEPTH}}$$

and

$$\boxed{L \leq CN \qquad (N = \#(E)).}$$

That's TODAY's THM.

It deals with FINITE E,

but

Existence of BOUNDED EXTENSION OPERATORS $T: L^{m,p}(E) \to L^{m,p}(\mathbb{R}^n)$ for ARBITRARY (infinite) E follows from TODAY's THM by considering arbitrary finite subsets of E and passing to a BANACH LIMIT.

The results stated last time amount to saying that

$\boxed{\text{TODAY'S THM HOLDS}}$

and that

$\Omega, \xi_1, \dots, \xi_L, T$

CAN BE COMPUTED EFFICIENTLY.

The Computation of

$\Omega, \xi_1, \ldots, \xi_L, T$

is a Long Story.

We won't get into that today (except to make a few brief remarks later).

INTRO TO THE PROOF OF TODAY'S THM.

IN MANY WAYS, OUR PROOF FOR $L^{m,p}$ CLOSELY FOLLOWS OUR PROOF FOR C^m (in the case of finite E).

WE WILL INTRODUCE

$\boxed{\text{LOCAL INTERPOLATION PROBLEMS}}$

and ATTACH

$\boxed{\text{LABELS}}$

TO SUCH PROBLEMS,

BASED ON THE SIZE & SHAPE OF A RELEVANT CONVEX SET

$\boxed{\sigma.}$

WE WILL SOLVE

LOCAL INTERPOLATION PROBS.

THAT CARRY A GIVEN

LABEL $\mathfrak{a}$,

BY USING A

$\boxed{\text{CALDERÓN-ZYGMUND DECOMPOSITION}}$

TO REDUCE MATTERS TO

SUBPROBLEMS THAT CARRY

EASIER LABELS $\mathfrak{a}' < \mathfrak{a}$.

However, there are

SIGNIFICANT DIFFERENCES

between our proofs for $L^{m,p}$ & our earlier proofs for C^m.

For C^m, we first computed

σ's & Γ's,

then USED THEM to

produce interpolants F.

For $L^{m,p}$, there's no

simple procedure to compute

σ's & Γ's.

(Will ultimately find them, but

only at the <u>end</u> of the story.)

Moreover, as far as I know,

We can't use T's

to study $L^{m,p}$ problems.

(We'll soon see why.)

Therefore, to prove

Today's Thm, we will need

<u>Essentially New Ideas</u>.

So our proof of TODAY'S THM

will look a lot like our

earlier proof for C^m,

until suddenly the C^m methods

CRASH

and we have to do something

COMPLETELY DIFFERENT.

LET'S DEFINE

σ's and Γ's

for $L^{m,p}$ PROBLEMS.

- We'll see why the Γ's can't be used, then we'll forget about them.

- The σ's WILL BE CRUCIAL.

Let $E \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$.

Then $\sigma(E, x)$ denotes the set of all $J_x(G)$ for fns. $G \in L^{m,p}(\mathbb{R}^n)$ such that

$$\|G\|_{L^{m,p}(\mathbb{R}^n)} \leq 1$$

and

$$G = 0 \text{ on } E.$$

Similarly, let $E \subset \mathbb{R}^n$, $f : E \to \mathbb{R}$, $x \in \mathbb{R}^n$, $M > 0$.

Then $\Gamma_f(x, M) \subset \mathcal{P}$ denotes the set of all $J_x(F)$ for fns. $F \in L^{m,p}(\mathbb{R}^n)$ such that

$$\|F\|_{L^{m,p}(\mathbb{R}^n)} \leq M$$

and

$$F = f \text{ on } E.$$

So $\sigma(E, x) \subset \mathcal{P}$

is a symmetric convex set,

and $\Gamma_f(x, M) \subset \mathcal{P}$

is a (possibly empty) convex set.

NOTE:

These are the TRUE σ and Γ.

WE DON'T HAVE USEFUL σ_ℓ, Γ_ℓ as in the C^m CASE.

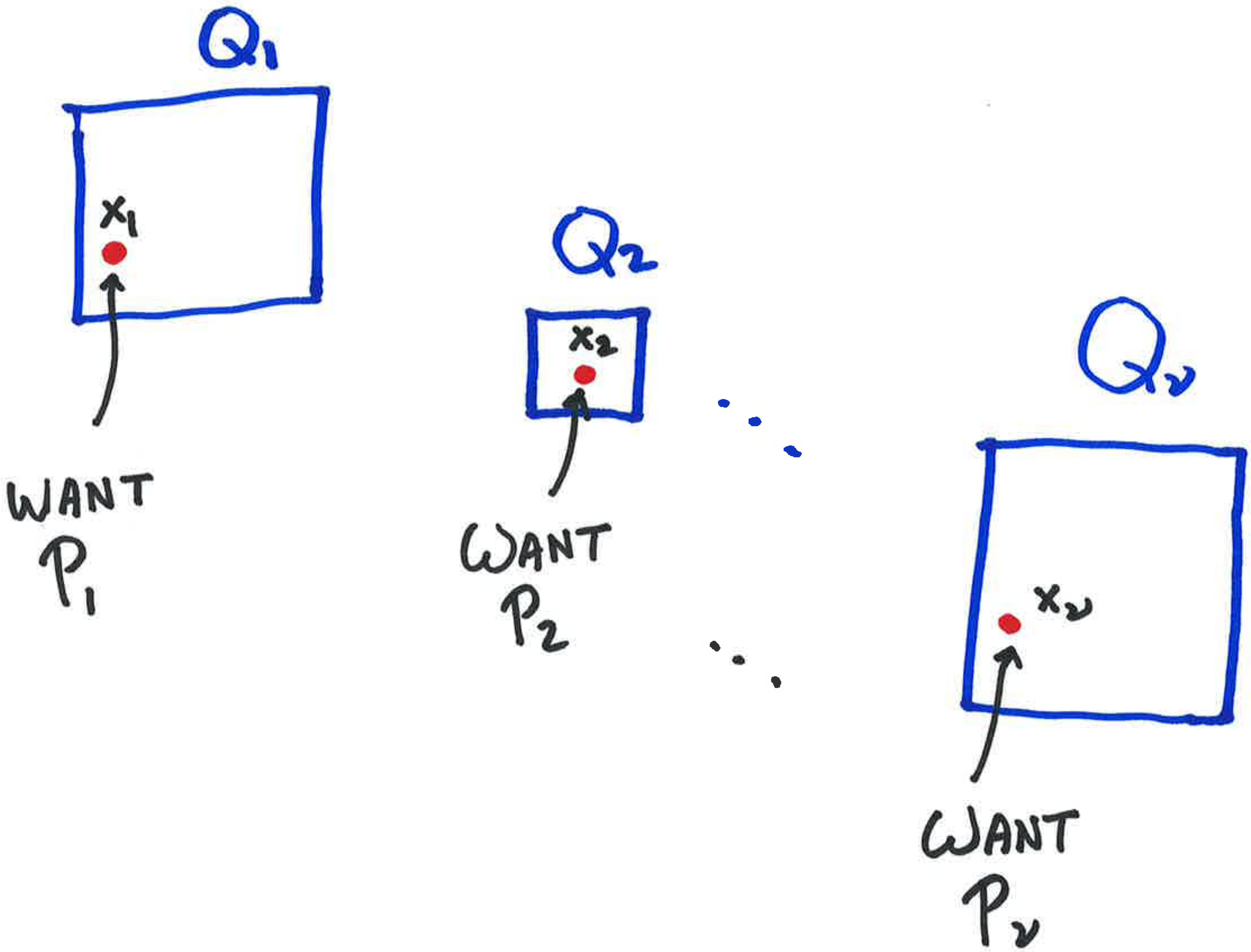
WHY THE Γ'S ARE (APPARENTLY) USELESS

We will construct an interpolant F by making a Calderón-Zygmund decomposition, finding a local interpolant F_Q for each $Q \in CZ$, and then patching together the F_Q.

To determine F_Q, we will have to associate a poly. $P_Q \in \mathcal{P}$ to a base pt. x_Q that lies in or near Q.

We have to do this for each CZ cube Q.

So far, it sounds familiar from the C^m case.



How to find the P_ν ?

By analogy with the C^m case, we might look for

$P_\nu \in \Gamma_f(x_\nu, M)$.

Then, for each ν, we could find $F_\nu \in L^{m,p}(\mathbb{R}^n)$ such that

$$\boxed{\begin{gathered} J_x(F_\nu) = P_\nu \quad \text{and} \\ \sum_{|\alpha|=m} \int_{\mathbb{R}^n} |\partial^\alpha F_\nu(x)|^p dx \le CM^p. \end{gathered}}$$

However, to achieve this, we might have to concentrate $\partial^\alpha F_\nu$ in or near Q_ν.

That would make it impossible to find a single $F \in L^{m,p}(\mathbb{R}^n)$ (indep. of ν) such that

$$\boxed{\begin{array}{l} J_{x_\nu}(F) = P_\nu \ \underline{\text{for all}}\ \nu, \quad \text{and} \\ \displaystyle\sum_{|\alpha|=m} \int_{\mathbb{R}^n} |\partial^\alpha F(x)|^p dx \le CM^p \end{array}}$$

That's because we can't concentrate $\partial^\alpha F$ mostly in Q_ν for EACH ν. Rather,

$$\boxed{\int_{\mathbb{R}^n} |\partial^\alpha F(x)|^p dx \text{ is a FINITE RESOURCE, that must be allocated among the } Q_\nu.}$$

That's a fundamental

difference between the

$L^{m,p}$ SEMINORM

(which is defined by an INTEGRAL)

and the

C^m NORM,

which is defined by a SUP.

WE NO LONGER DISCUSS THE Γ'S.

LET'S RETURN TO THE PROOF OF TODAY'S THM.

WE NEXT PREPARE TO DEFINE

LOCAL INTERPOLATION PROBLEMS.

Let $E \subset \mathbb{R}^n$ (finite), $z \in \mathbb{R}^n$.

Given $f : E \to \mathbb{R}$ and $P \in \mathcal{P}$,

we define

$$\|(f, P)\|_{E,z}$$

as the inf of $\|F\|_{L^{m,p}(\mathbb{R}^n)}$

over all $F \in L^{m,p}(\mathbb{R}^n)$ such that

$$\boxed{F = f \text{ on } E \quad \text{and} \quad J_z(F) = P.}$$

An EXTENSION OPERATOR for (E, z) is a linear map T, sending each (f, P) as above, to a fn. $F = T(f, P)$ satisfying $F = f$ on E and $J_z(F) = P$.

An extension operator T for (E, z) is called <u>C-optimal</u> if it satisfies

$$\| T(f, P) \|_{L^{m,p}(\mathbb{R}^n)} \leq C \| (f, P) \|_{E, z}$$

for all (f, P).

LET Ω BE A SET OF ASSISTS FOR E.

A linear functional

$$\xi : L^{m,p}(E) \oplus \mathcal{P} \longrightarrow \mathbb{R}$$

or a linear map

$$T : L^{m,p}(E) \oplus \mathcal{P} \to L^{m,p}(\mathbb{R}^n)$$

MAY HAVE

Ω-ASSISTED BDD. DEPTH.

(WE LEAVE THE DEFINITION

TO YOUR IMAGINATION.)

Now we can define

LOCAL INTERPOLATION

PROBLEMS.

Given a cube $Q \subset \mathbb{R}^n$,

a finite set $E \subset Q$,

and a point $z \in Q$,

we want to find the following:

$\boxed{\text{A SET } \Omega \text{ OF ASSISTS for } E.}$

A C-OPTIMAL LINEAR EXTENSION OPERATOR $T: \mathbb{L}^{m,p}(E) \oplus \mathcal{P} \to \mathbb{L}^{m,p}(\mathbb{R}^n)$ OF Ω-ASSISTED BDD DEPTH

Also, ...

(A FORMULA FOR THE NORM)

LINEAR FUNCTIONALS

$$\xi_1, \dots, \xi_L : L^{m,p}(E) \oplus \mathcal{P} \to \mathbb{R}$$

such that

Each ξ_ℓ is of Ω- ASSISTED BDD DEPTH

$$L \leq C \cdot \#(E)$$

$$\|(f, P)\|^p_{E,z} \sim \sum_{\ell=1}^{L} |\xi_\ell(f, P)|^p$$

for any $(f, P) \in L^{m,p}(E) \oplus \mathcal{P}$

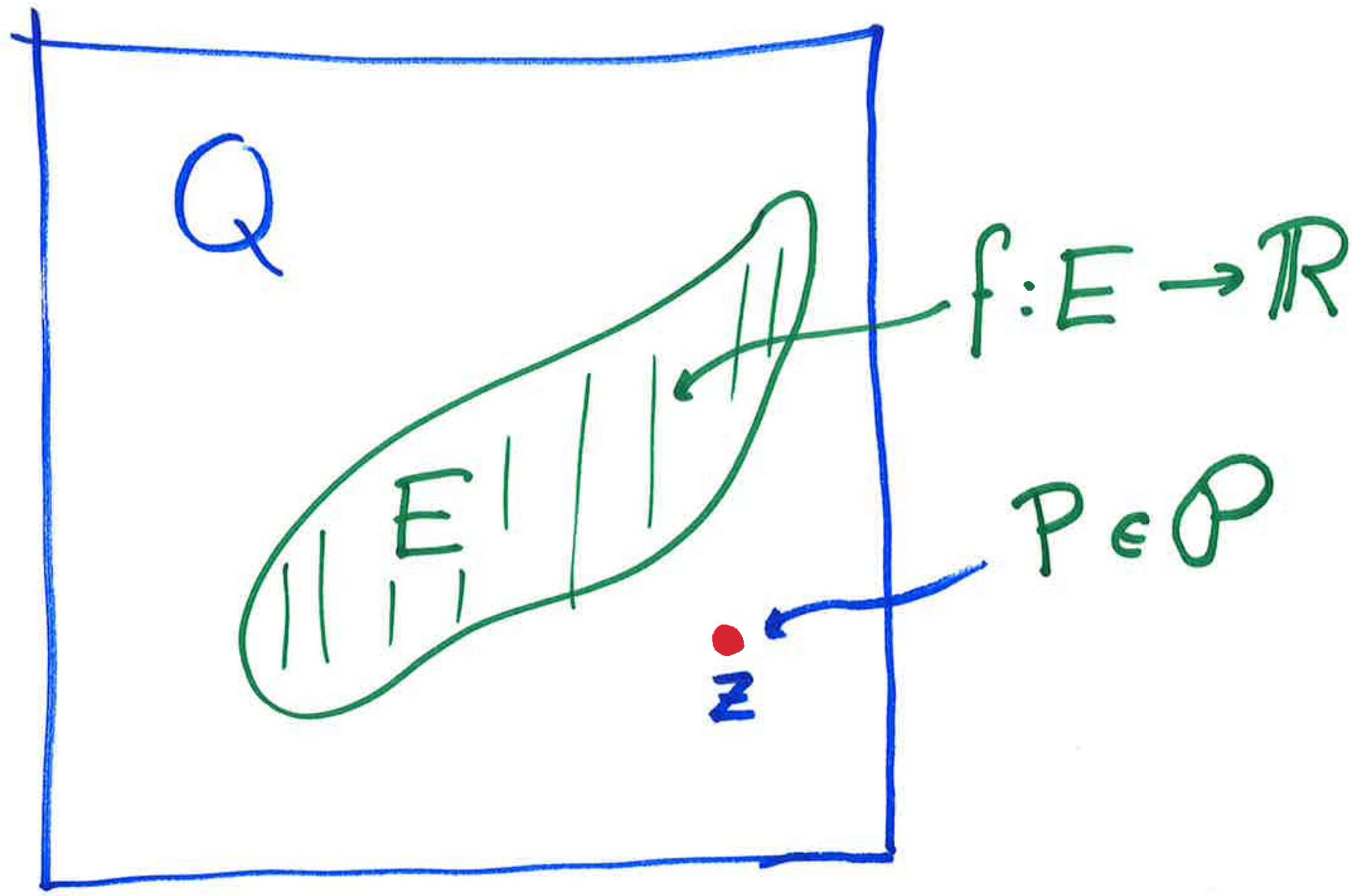
The Local Interpolation Problem

$\mathrm{LIP}(Q, E, z)$

is the problem of producing

Ω, T, $\xi_1, \dots, \xi_L$

as above.



WANT: • Ω, SET OF ASSISTS

• T linear, Ω-ABD s.t.

$F = T(f, P)$ SATISFIES

$F = f$ on E, $J_z(F) = P$,

SEMINORM $\|F\|$ minimized up to C

• FORMULA FOR $\|F\|$.

TODAY'S THM

follows easily

if we can show that

$$\boxed{\text{EVERY } LIP(Q, E, z) \text{ HAS A SOLUTION.}}$$

To solve $LIP(Q, E, z)$,

WE ATTACH LABELS.

As in the C^m Case, a label is a subset $\mathcal{A}$ of

$$\mathcal{M} = \{ \text{MULTI-INDICES } \alpha \text{ OF ORDER } |\alpha| \leq m-1 \}.$$

Labels are ordered by $<$ as in the C^m case.

(ALMOST) as in the C^m CASE,

WE ATTACH A GIVEN LABEL $\mathcal{A}$ TO A GIVEN LIP(Q, E, z)

$\Longleftrightarrow$

$\sigma(E, z)$ has an $(\mathcal{A}, \hat{C}(\mathcal{A})\delta_Q)$-basis at z.

Here, • δ_Q = SIDELENGTH(Q)

• $\hat{C}(\mathcal{A})$ = LARGE CONST. DEP. ONLY ON $\mathcal{A}$, m, n, p

• DEF. of $(\mathcal{A}, \delta)$-BASIS TBA

NOTE:

HERE WE USE THE TRUE $\sigma(E, z)$.

RECALL THAT FOR C^m PROBLEMS, WE USED $\sigma_\ell(z)$, $\ell = \ell(Q)$.

For $L^{m,p}$ there are no σ_ℓ.

An (Q, δ_Q)-BASIS has a slightly different meaning here, as compared to the setting of $C^m(\mathbb{R}^n)$,

simply because $L^{m,p}(\mathbb{R}^n)$ and $C^m(\mathbb{R}^n)$ scale differently.

DEFINITION:

An $(\mathcal{Q}, \delta)$-BASIS FOR

a symmetric convex SET $\sigma \subset \mathcal{P}$

at a point $z \in \mathbb{R}^n$

is a family $(P_\alpha)_{\alpha \in \mathcal{Q}}$

of polys. $P_\alpha \in \mathcal{P}$,

WITH THE FOLLOWING PROPERTIES:

$$\partial^\beta P_\alpha(z) = \delta_{\beta\alpha} \quad [\text{KRONECKER}]$$

for $\beta, \alpha \in \mathcal{A}$

$$|\partial^\beta P_\alpha(z)| \leq C_\alpha(\mathcal{A}) \cdot \delta^{|\alpha| - |\beta|}$$

for $\alpha \in \mathcal{A}$, $|\beta| \leq m-1$.

$$\delta^{m - |\alpha| - \frac{n}{p}} P_\alpha \in C_\alpha(\mathcal{A})\,\sigma$$

for $\alpha \in \mathcal{A}$.

NOTE THE $-\frac{n}{p}$ IN THE EXPONENT

IN THE ABOVE DEFINITION, $C(a)$ IS A LARGE CONSTANT DETERMINED BY a, m, n, p.

SO NOW WE HAVE DEFINED AN $(\mathcal{A}, \delta)$-BASIS.

RECALL:

$$\boxed{\begin{array}{l}\text{We attach label } \mathcal{A} \text{ to } LIP(Q,E,z) \\ \Longleftrightarrow \\ \sigma(E,z) \text{ has an} \\ (\mathcal{A}, \hat{C}(\mathcal{A})\delta_Q)\text{-basis at } z.\end{array}}$$

By induction on $\mathcal{a}$ (with respect to $<$),

we PROVE THE

MAIN LEMMA FOR $\mathcal{a}$:

Any $LIP(Q, E, z)$ that carries the label $\mathcal{a}$ can be solved.

As in the C^m case,

EVERY LIP(...)

CARRIES THE LABEL $\emptyset$ (EMPTY SET).

Therefore,

[MAIN LEMMA for $\emptyset$]

$\Downarrow$

[ANY LIP(Q, E, z) CAN BE SOLVED]

$\Downarrow$

[TODAY'S THM]

THE INDUCTION on a

BASE CASE: $a = m$.

LIP(Q, E, z) CARRIES LABEL m

$\iff E = \emptyset$.

So WE CAN SOLVE LIP(Q, E, z)

by taking $T(P) = P$. (No f!)

The fn P satisfies all requirements.

NO ASSISTS NEEDED.

INDUCTION STEP

Fix $a \neq m$, and ASSUME THE

INDUCTION HYPOTHESIS:
The MAIN LEMMA for a' holds for all $a' < a$.

Under that ASSUMPTION, we must prove the MAIN LEMMA for a.

That is,

$$\boxed{\begin{array}{l}\text{WE MUST SOLVE EVERY} \\ \text{LIP}\,(Q^\circ, E, z^\circ) \\ \text{THAT CARRIES} \\ \text{THE LABEL } a.\end{array}}$$

RECALL, A SUBSET $\mathcal{A} \subset \mathcal{M}$

IS MONOTONIC

IF FOR EVERY $\alpha \in \mathcal{A}$

AND EVERY $\gamma \in \mathcal{M}$ of order

$|\gamma| \leq m-1-|\alpha|$,

WE HAVE $\alpha + \gamma \in \mathcal{M}$.

IF $\mathcal{A}$ IS NOT MONOTONIC,

THEN

$$\boxed{\mathcal{A}' \equiv \{\alpha+\gamma : \alpha \in \mathcal{A},\ |\gamma| \leq m-1-\alpha\} < \mathcal{A},}$$

AND

$$\boxed{\text{EVERY LIP}(\ldots)\text{ THAT CARRIES THE LABEL } \mathcal{A} \text{ ALSO CARRIES THE LABEL } \mathcal{A}'.}$$

THEREFORE,

THE INDUCTION STEP

IN OUR PROOF OF THE

MAIN LEMMA for $\mathcal{a}$

IS TRIVIAL for

NON-MONOTONIC $\mathcal{a}$.

FROM NOW ON,

WE SUPPOSE

THAT a IS

MONOTONIC.

SO OUR SETTING IS AS FOLLOWS:

We fix a MONOTONIC LABEL $a < m$.

We assume THE MAIN LEMMA for all $a' < a$.

We fix a $LIP(Q^\circ, E, z^\circ)$ THAT CARRIES THE LABEL a.

Our task is to solve

$\mathrm{LIP}(\mathcal{Q}^\circ, E, z^\circ)$.

Once we do so, our induction on $\mathcal{A}$ is complete, thus proving the MAIN LEMMA for all $\mathcal{A}$, and proving TODAY'S THM.

RECALL THE PROBLEM

$LIP(Q^\circ, E, z^\circ)$.

WE MUST PRODUCE A SET

OF ASSISTS Ω,

A LINEAR MAP

$T : L^{m,p}(E) \oplus \mathcal{P} \rightarrow L^{m,p}(\mathbb{R}^n)$,

and FUNCTIONALS $\xi_1, \dots, \xi_L$

on $L^{m,p}(E) \oplus \mathcal{P}$, SUCH THAT:

(1) For any $(f, P) \in L^{m,p}(E) \oplus \mathcal{P}$,

the function

$F = T(f, P) \in L^{m,p}(\mathbb{R}^n)$

satisfies

$F = f$ on E and $J_{z^0}(F) = P$,

with $\|F\|_{L^{m,p}(\mathbb{R}^n)}$

as small as possible

up to a factor C.

(2)

For any $(f, P) \in L^{m,p}(E) \oplus \mathcal{P}$,

we have

$$\boxed{\begin{aligned} c \cdot \sum_{\ell=1}^{L} |\xi_\ell(f, P)|^p &\le \|(f, P)\|_{E, z^0} \\ &\le C \cdot \sum_{\ell=1}^{L} |\xi_\ell(f,P)|^p \end{aligned}}$$

③ $\boxed{\text{The functionals } \xi_1, \ldots, \xi_L \text{ and the operator } T \text{ are of } \Omega\text{-assisted Bdd. Depth.}}$

④ $\boxed{L \leq CN \qquad (N = \#(E)).}$

THAT'S OUR PROBLEM.

TO SOLVE IT,

WE MAKE A

CALDERÓN - ZYGMUND

DECOMPOSITION OF

$3Q^{\circ}$.

STARTING WITH 30°,

WE REPEATEDLY BISECT,

UNTIL THE FOLLOWING

RULE TELLS US TO STOP.

STOPPING RULE

STOP CUTTING Q

WHEN EITHER

$$\boxed{\#(E \cap 3Q) \leq 1}$$

or

There exists a label $a' < a$ such that $LIP(3Q, E \cap 3Q, x)$ CARRIES THE LABEL a' FOR ALL $x \in 3Q$.

This procedure partitions $3Q^{\circ}$ into finitely many CZ cubes.

For each $Q \in CZ$,

we pick a base point

$z_Q \in Q$, NOT TOO CLOSE TO E.

(Can do this, else our CZ rule would tell us to bisect Q.)

Our PLAN is to associate

to each z_Q a poly. $P_Q \in \mathcal{P}$.

We then define local interpolants

$F_Q \in L^{m,p}(\mathbb{R}^n)$ satisfying

$$\boxed{\begin{array}{l} F_Q = f \text{ on } E \cap 3Q \\ \text{and} \\ J_{z_Q}(F_Q) = P_Q \end{array}}$$

with $\|F_Q\|_{L^{m,p}(\mathbb{R}^n)}$ controlled

We then define our interpolant F by setting

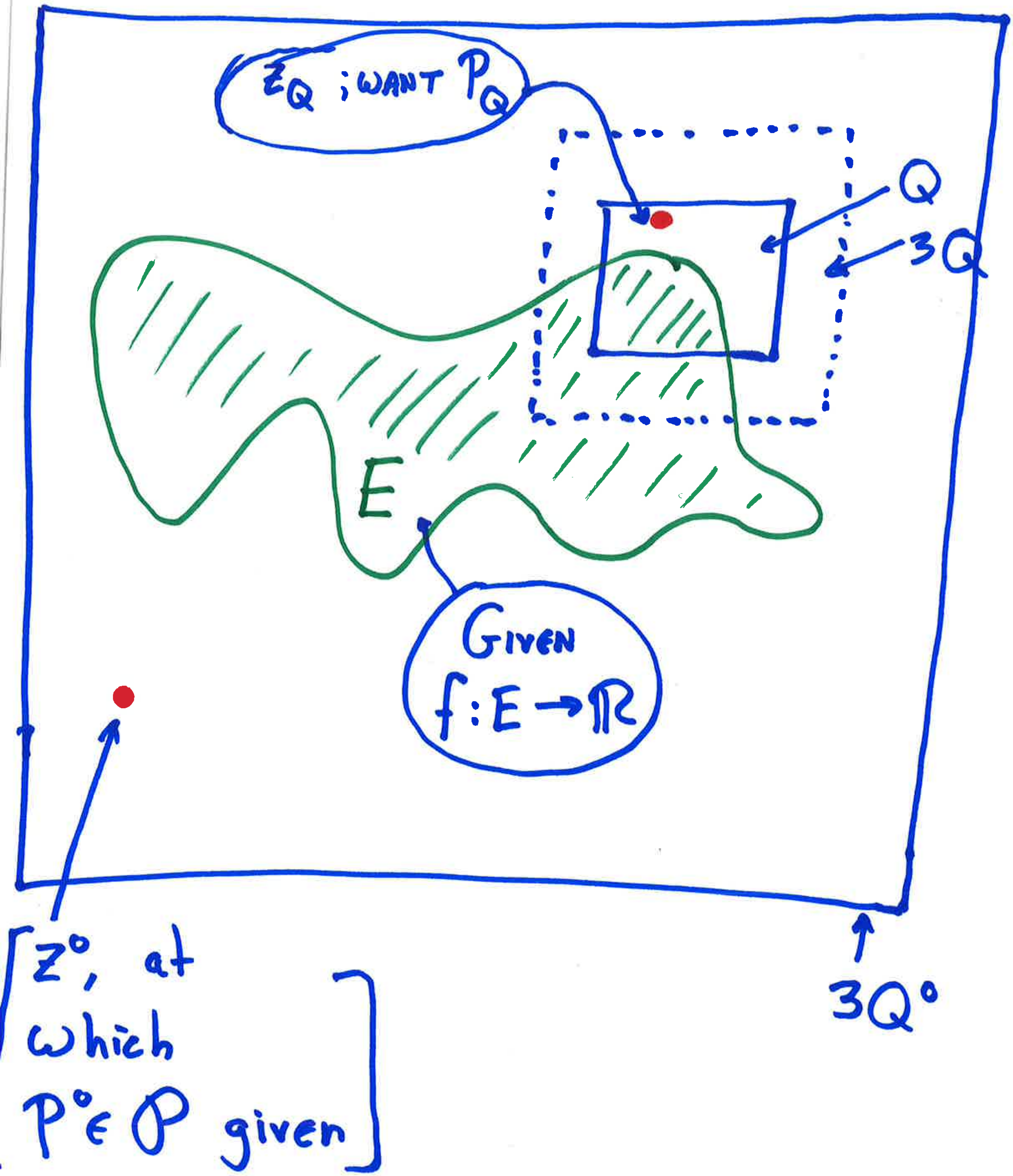
$$F = \sum_Q \theta_Q F_Q ,$$

using a Whitney partition of unity $\{\theta_Q\}$ associated to the CZ decomposition.

We hope that:

- $\|F\|_{L^{m,p}(\mathbb{R}^n)}$ is controlled.
- F depends on (f, P) via a linear operator of ABD.
- There is a formula for $\|F\|_{L^{m,p}}$.

THAT'S WHAT WE NEED OUR F TO DO.


z_Q ; WANT P_Q
Q
3Q
E
GIVEN
$f : E \to \mathbb{R}$
3Q°
[z°, at which $P^\circ \in \mathcal{P}$ given]

FINDING THE F_Q

For each CZ cube Q,

EITHER

$\#(E \cap 3Q) \leq 1$

(SO IT'S TRIVIAL TO FIND F_Q)

OR . . .

THE LOCAL INTERPOLATION PROBLEM

$LIP(3Q, E \cap 3Q, z_Q)$

CARRIES A LABEL

$a' < a$,

thanks to our

STOPPING RULE.

IN THAT CASE,

WE CAN FIND A SUITABLE

F_Q, THANKS TO OUR

INDUCTION HYPOTHESIS

(MAIN LEMMA holds

for $a' < a$.)

We know that

$\boxed{\| F_Q \|_{L^{m,p}} \text{ is CONTROLLED}}$

$\boxed{\begin{array}{l} F_Q \text{ DEPENDS ON } (f|_{E \cap 3Q}, P_Q) \\ \text{VIA A LINEAR OPERATOR} \\ \text{OF ASSISTED BDD. DEPTH.} \end{array}}$

$\boxed{\begin{array}{l} \text{There's a FORMULA for} \\ \| F_Q \|_{L^{m,p}} . \end{array}}$

CAN WE USE

THAT INFO.

TO PROVE

THE DESIRED

PROPERTIES OF

$$F = \sum_Q \theta_Q F_Q \quad ?$$

As in the C^m case, this idea will work, provided we pick the P_Q wisely.

We must make sure that

Each P_Q depends on (f, P) via a linear operator of ABD

and

The P_Q are MUTUALLY CONSISTENT.

MUTUAL CONSISTENCY

OF THE P_Q

ENTERS BECAUSE WE

WILL USE THE

FOLLOWING

ELEMENTARY RESULT:

PATCHING LEMMA:

Suppose $F = \sum_Q \theta_Q F_Q$.

Let $z_Q \in Q$ and let $P_Q = J_{z_Q}(F_Q)$.

Then

$$\boxed{\|F\|^p_{L^{m,p}} \leq C\sum_Q \|F_Q\|^p_{L^{m,p}} + C\sum_{Q \leftrightarrow Q'} \delta_Q^{-mp} \|P_Q - P_{Q'}\|^p_{L^p(Q)}}$$

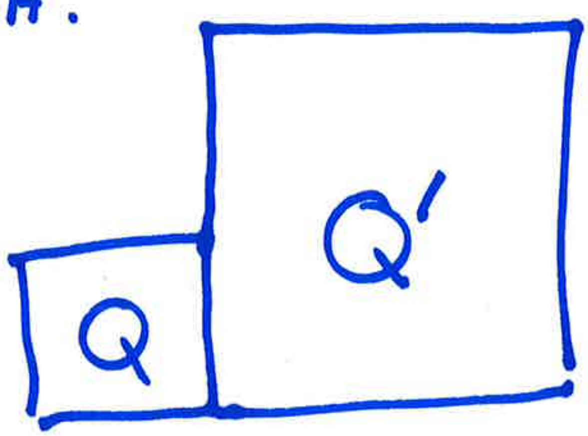
As usual,

δ_Q = SIDELENGTH (Q).

We write $Q \leftrightarrow Q'$

to denote that Q & Q'

touch.



SO FAR, EVERYTHING

LOOKS FAMILIAR

FROM THE C^m CASE.

THAT'S ABOUT TO STOP.

In the C^m case,

we pick the P_Q

using the convex sets Γ_ℓ.

In the Sobolev case,

there are no Γ_ℓ, and

we can't use Γ_f,

for reasons explained earlier.

WE NEED A NEW IDEA.

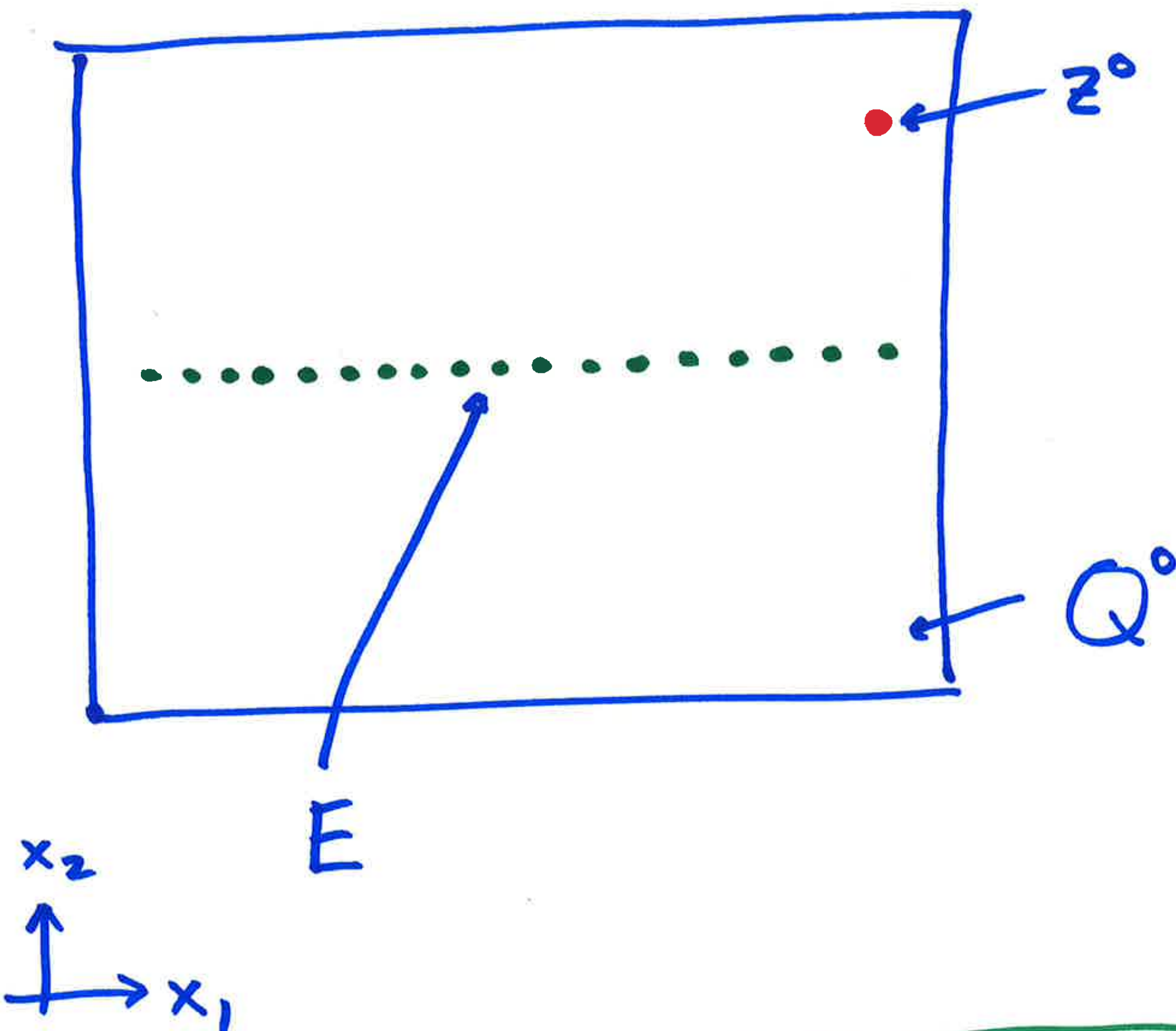
TO SEE WHAT TO DO,

WE LOOK AT

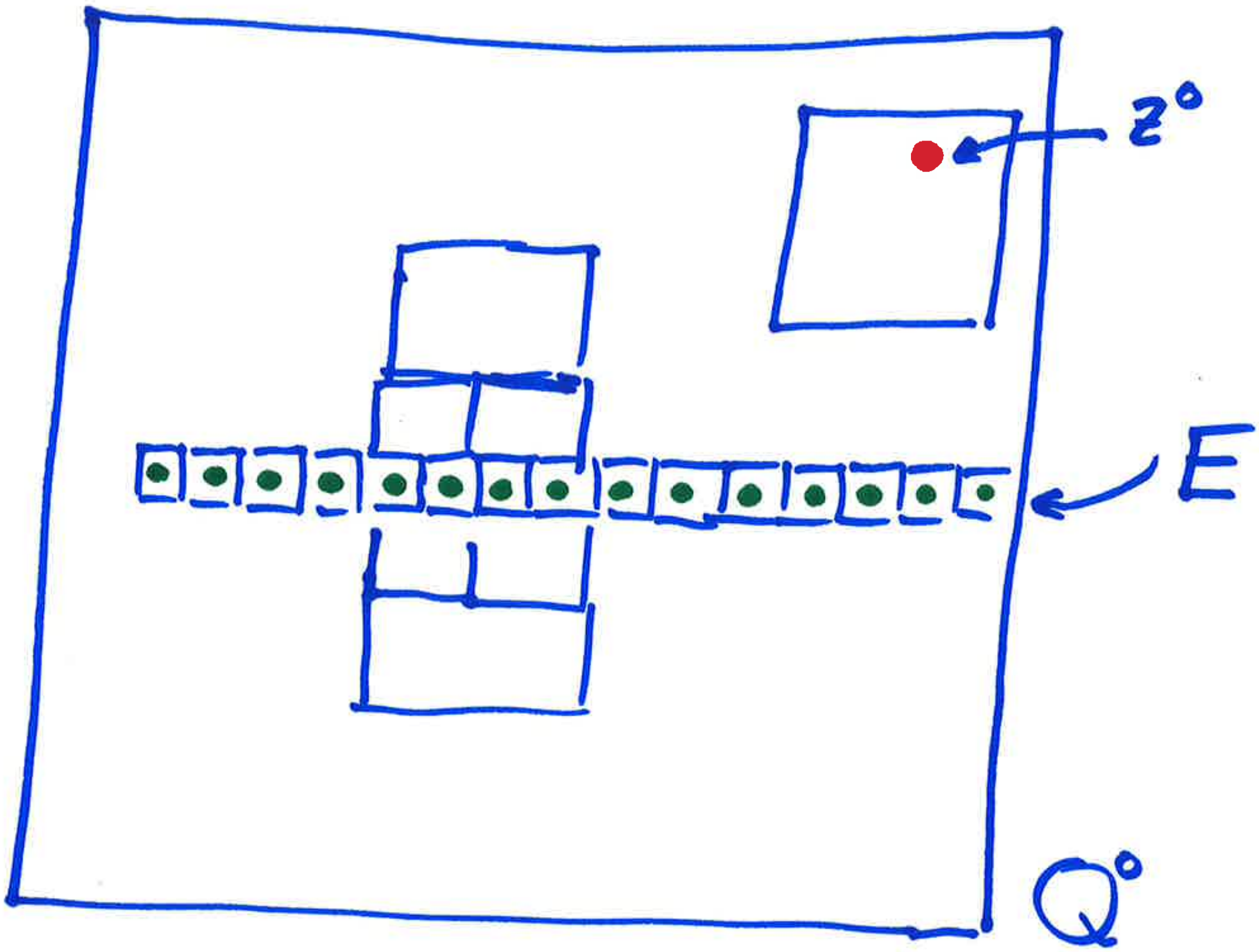
3 SIMPLE EXAMPLES

EXAMPLE 1

Work in $L^{2,p}(\mathbb{R}^2)$, $p > 2$.

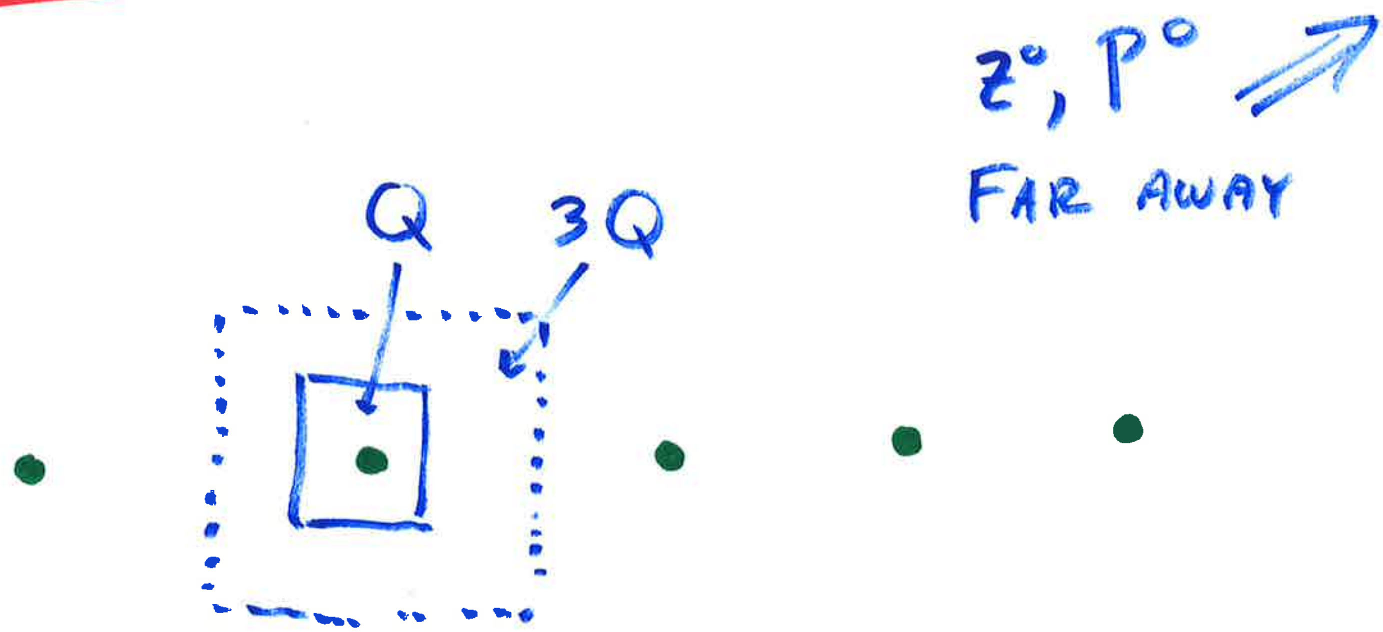


$$\boxed{\text{The LABEL is } \mathcal{A} = \{(0,1)\} \subset \mathcal{M}}$$



THE CALDERÓN-ZYGMUND CUBES IN EXAMPLE 1

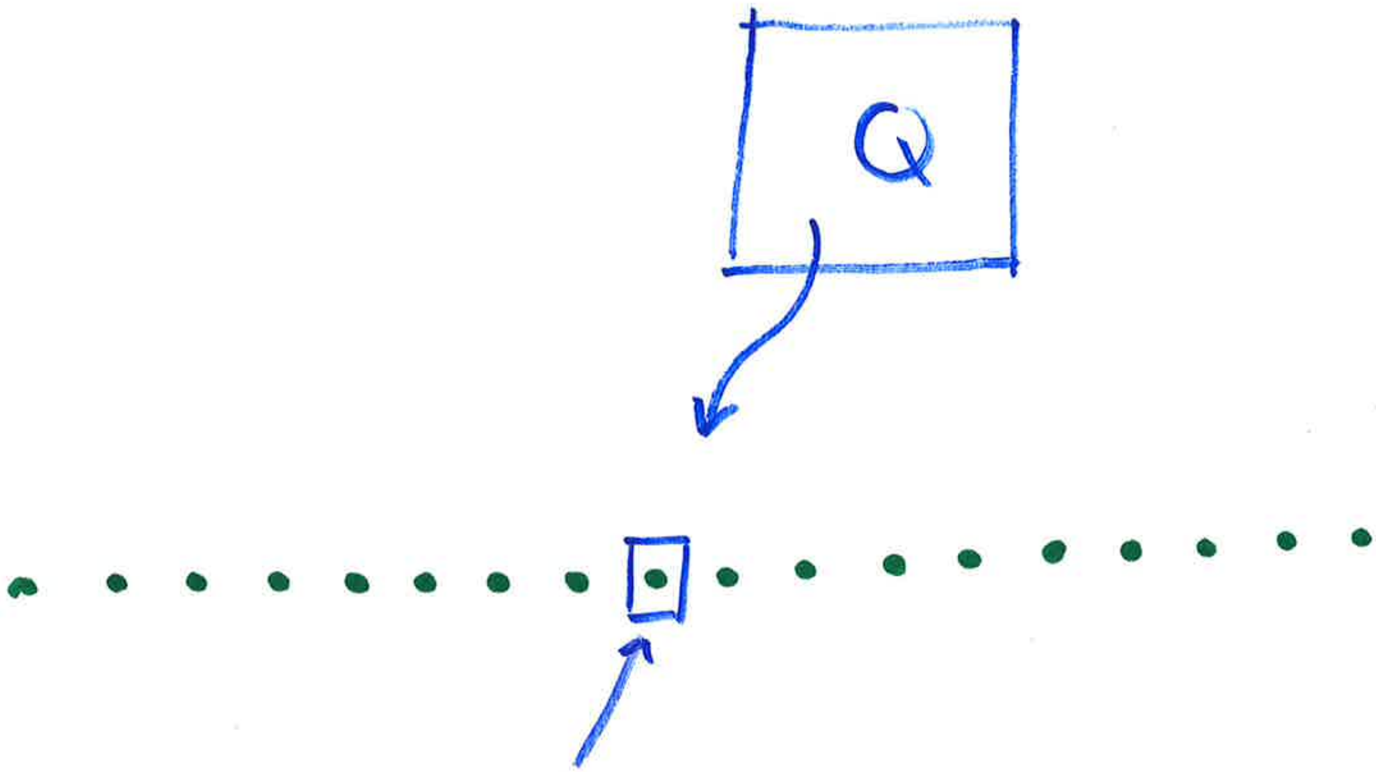
LOCAL INTERPOLANTS FOR THE SMALLEST CZ CUBES Q



DEFINE $F_Q\big|_{x_1\text{-axis}}$, e.g. by a SPLINE.

LET $\partial_{x_2} F_Q \equiv \partial_{x_2} P^\circ$.

Local Interpolants F_Q for Larger CZ cubes Q



$Q^{\#} \ni z^{\#}(Q^{\#})$

$$\boxed{\text{Set } F_Q := J_{z^{\#}(Q^{\#})}(F_{Q^{\#}}).}$$

One Exception:

If Q is the CZ cube

containing the BASE pt. z^0,

then just set

$$\boxed{F_Q = P^0.}$$

We have now defined F_Q for <u>all</u> the CZ cubes Q.

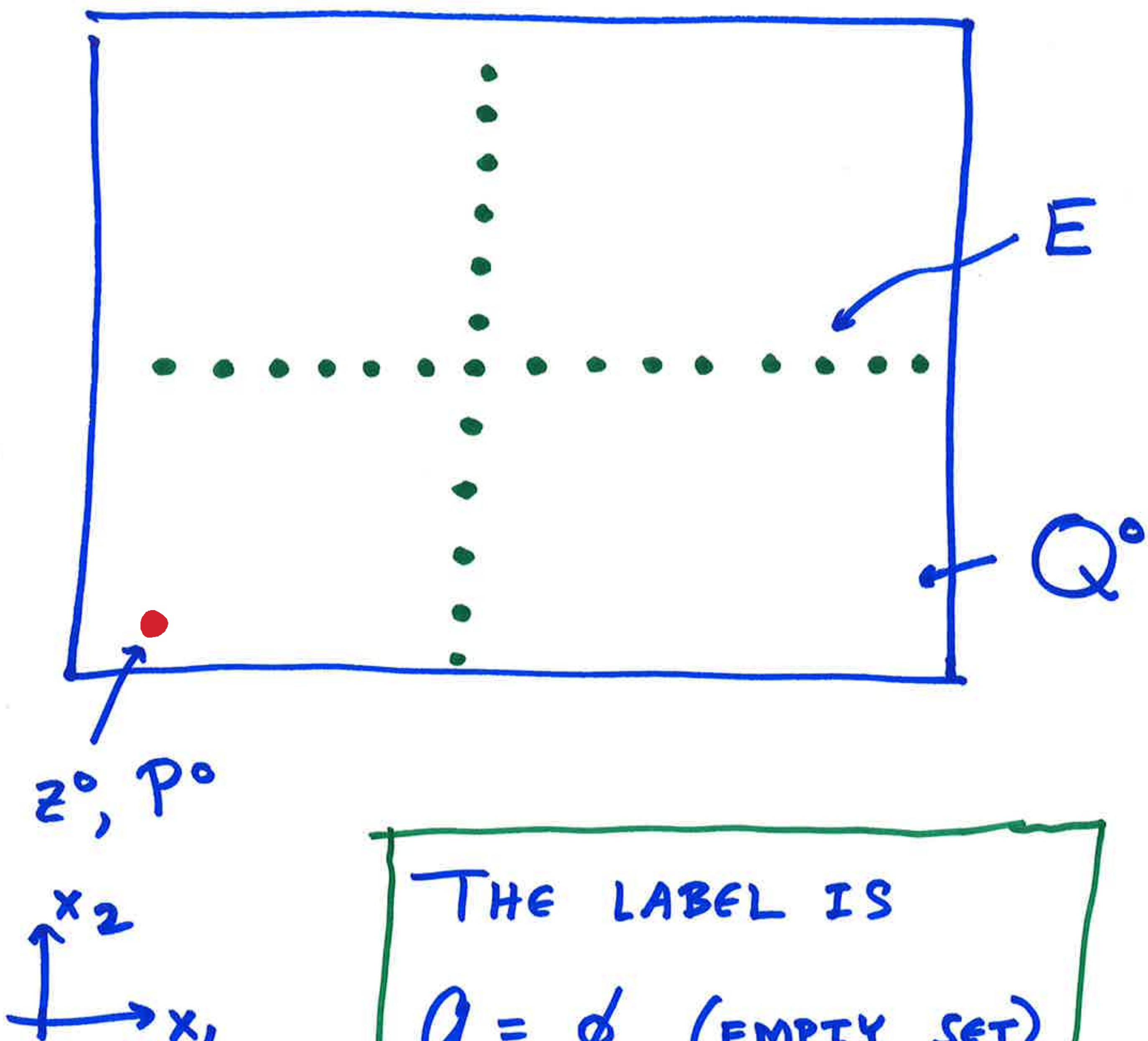
It works!

$$F = \sum_Q \theta_Q F_Q$$ has all the desired properties.

WE'RE DONE WITH EXAMPLE 1.

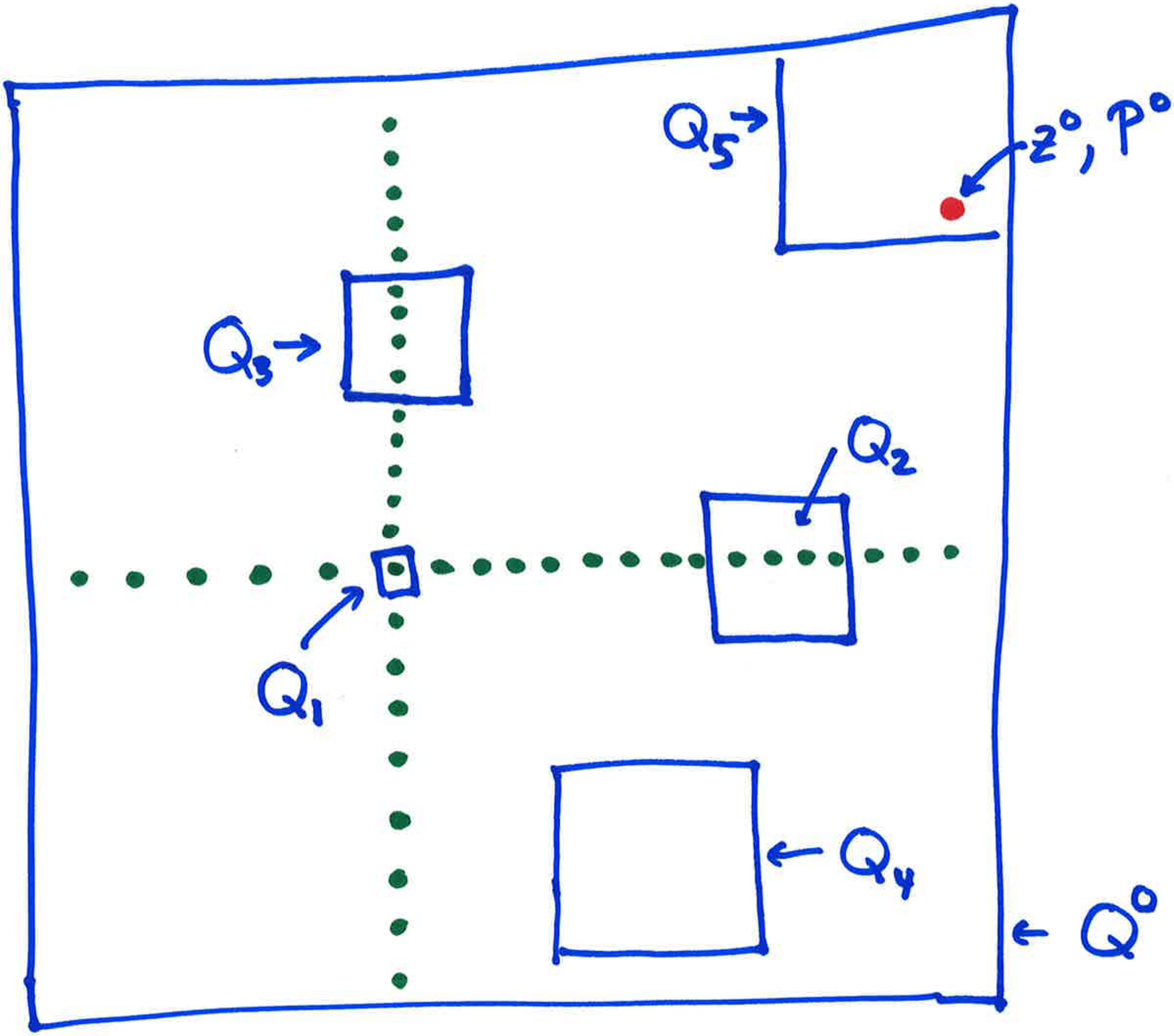
EXAMPLE 2

AGAIN WORK IN $L^{2,p}(\mathbb{R}^2)$, $p > 2$.



$$\boxed{\text{THE LABEL IS } \quad \mathcal{A} = \emptyset \text{ (EMPTY SET)}}$$

THE CZ CUBES FOR EXAMPLE 2

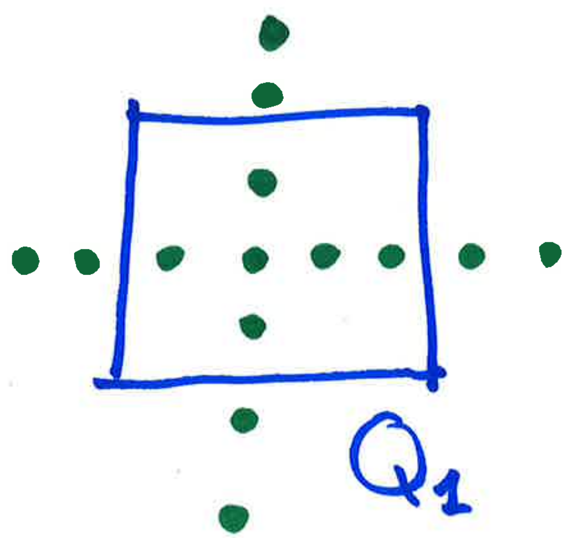


AT THE SMALLEST LENGTHSCALE,

THE CZ CUBE Q_1

GIVES RISE TO F_{Q_1}.

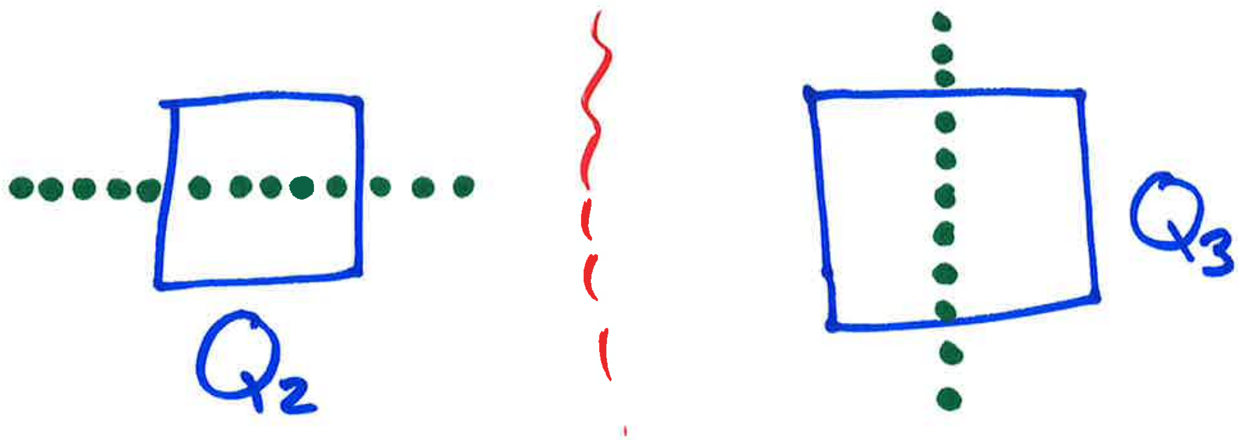
THERE IS NO SIGNIFICANT AMBIGUITY IN F_{Q_1}.



AT A LARGER LENGTHSCALE,

WE HAVE CZ CUBES

LIKE Q_2 & Q_3.



F_{Q_2} & F_{Q_3} HAVE SIGNIFICANT AMBIGUITY IF WE VIEW Q_2 or Q_3 IN ISOLATION.

HOWEVER, WE CAN SET

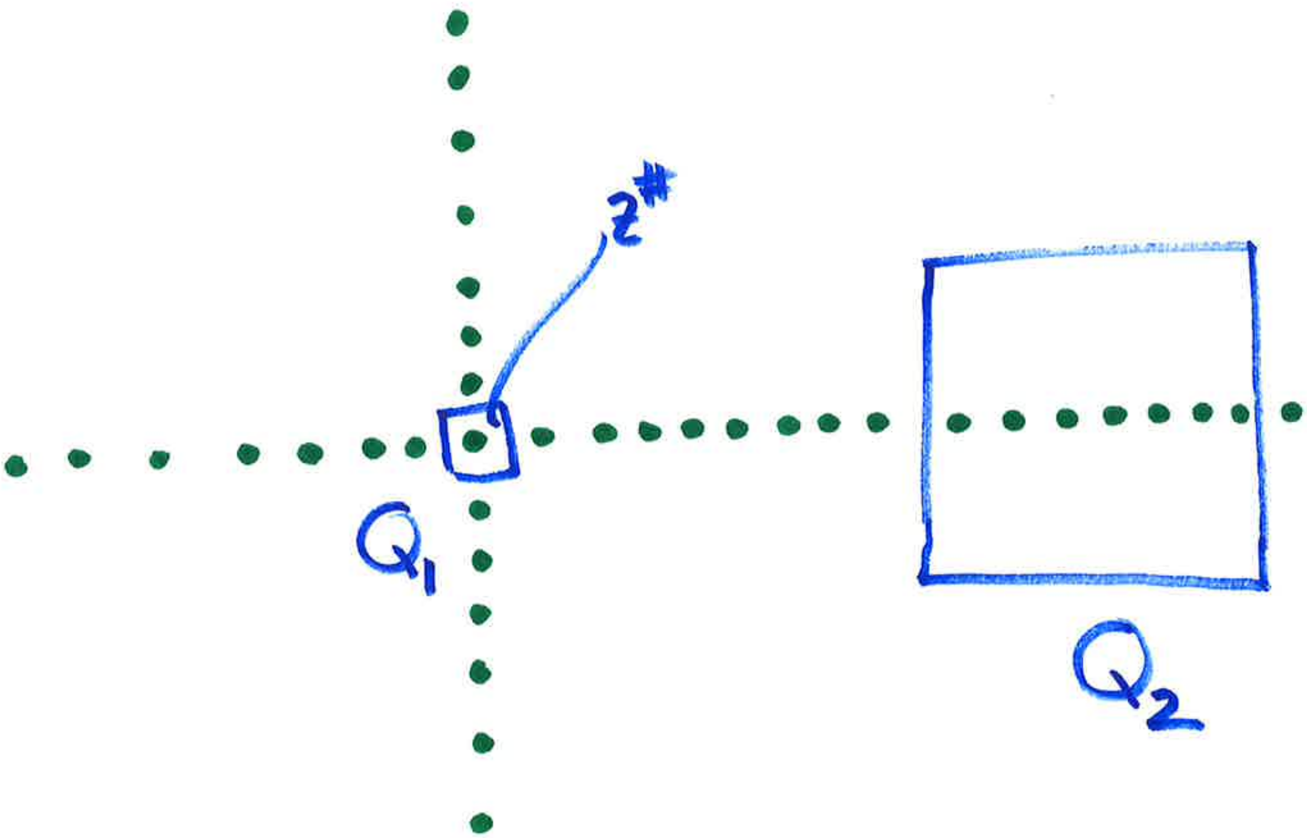
$$\boxed{\frac{\partial F_{Q_2}}{\partial x_2} \equiv \frac{\partial P^{\#}}{\partial x_2}}$$

and

$$\boxed{\frac{\partial F_{Q_3}}{\partial x_1} \equiv \frac{\partial P^{\#}}{\partial x_1}}$$

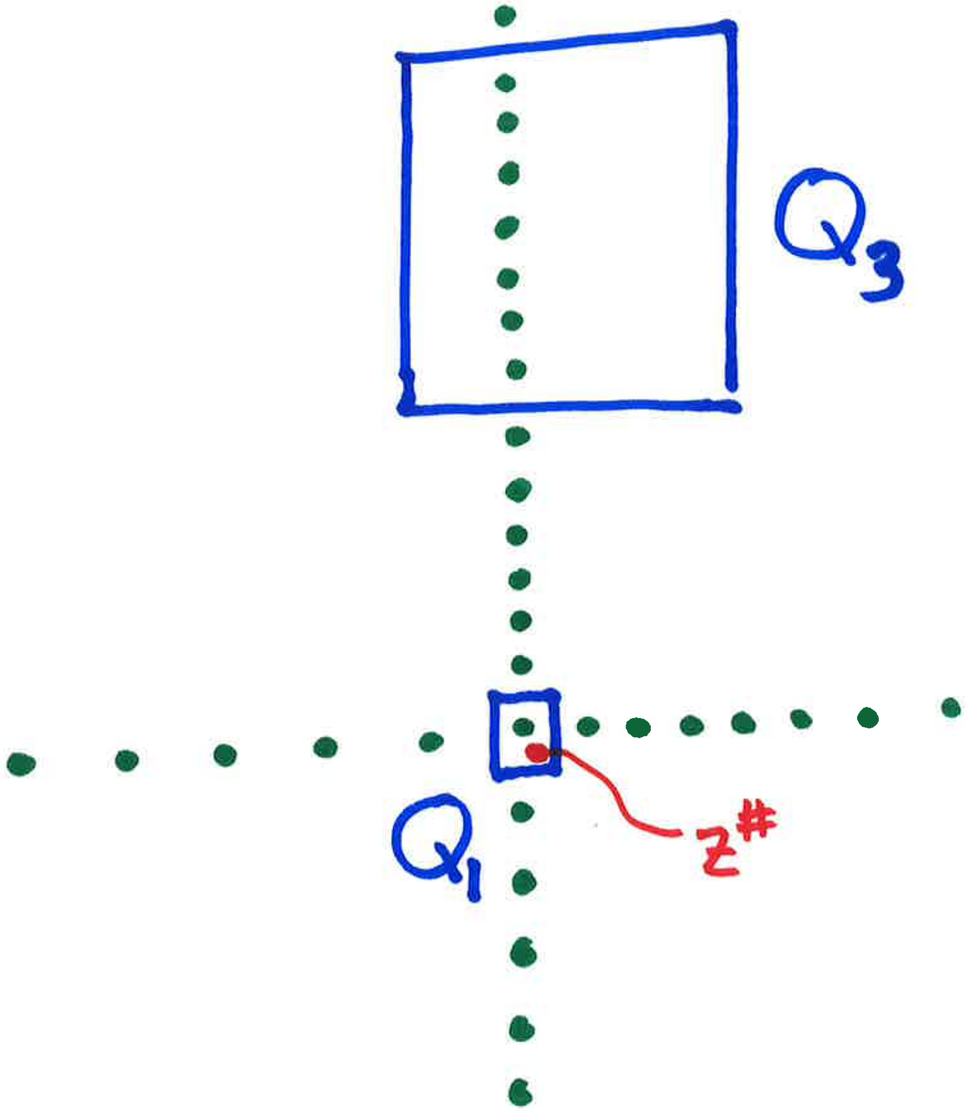
where $P^{\#} = J_{z^{\#}}(F_{Q_1})$

for some $z^{\#} \in Q_1$.



SET $$\frac{\partial F_{Q_2}}{\partial x_2} = \frac{\partial P^\#}{\partial x_2}$$

$$P^\# = J_{z^\#}(F_{Q_1})$$



SET $\dfrac{\partial F_{Q_3}}{\partial x_1} = \dfrac{\partial P^{\#}}{\partial x_1}$

where $P^{\#} = J_{z^{\#}}(F_{Q_1})$.

ONCE WE HAVE RESOLVED

THEIR SIGNIFICANT

AMBIGUITY AS EXPLAINED

JUST ABOVE,

WE CAN PRODUCE F_{Q_2} & F_{Q_3}

BY THE METHOD OF

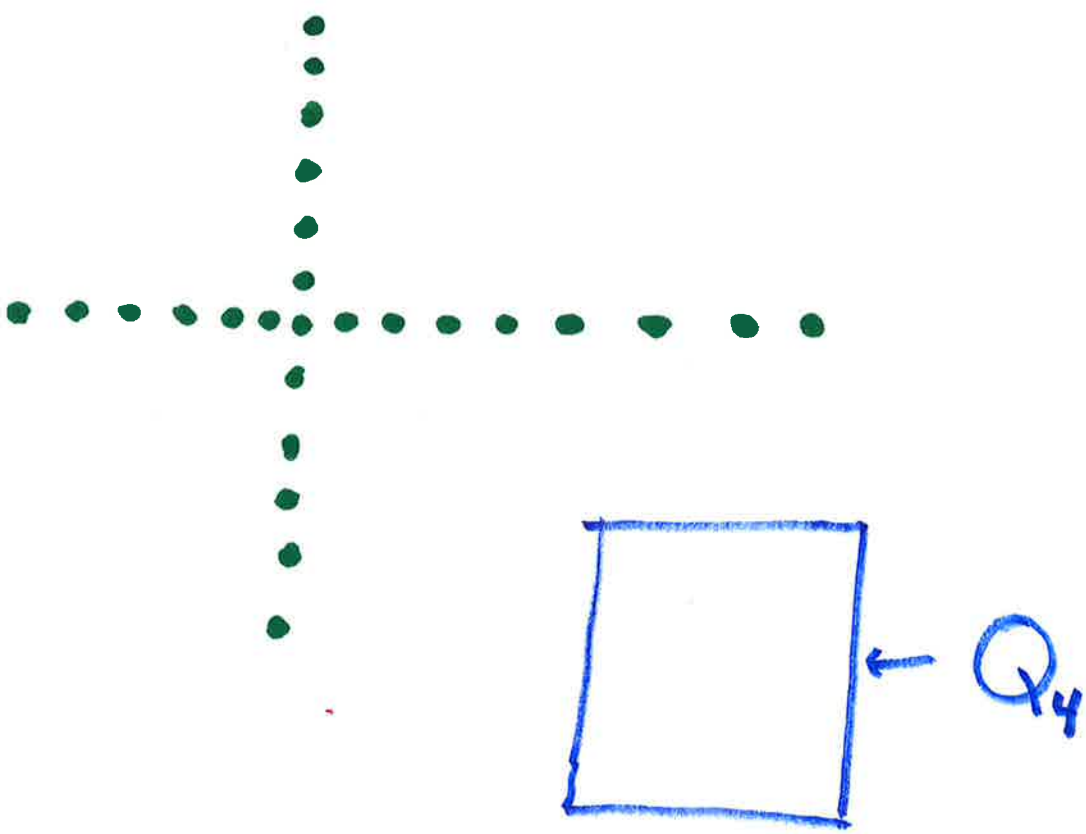
EXAMPLE 1.

For a cube such as Q_4,

we can just set

$$F_{Q_4} = \mathcal{P}^{\#}$$

(SAME $\mathcal{P}^{\#}$ AS ABOVE)



Finally, for the SPECIAL Q_5

that contains the

BASE POINT z^0,

WE JUST SET

$$F_{Q_5} = P^0.$$

(RECALL, z^0, P^0 GIVEN)

It works!

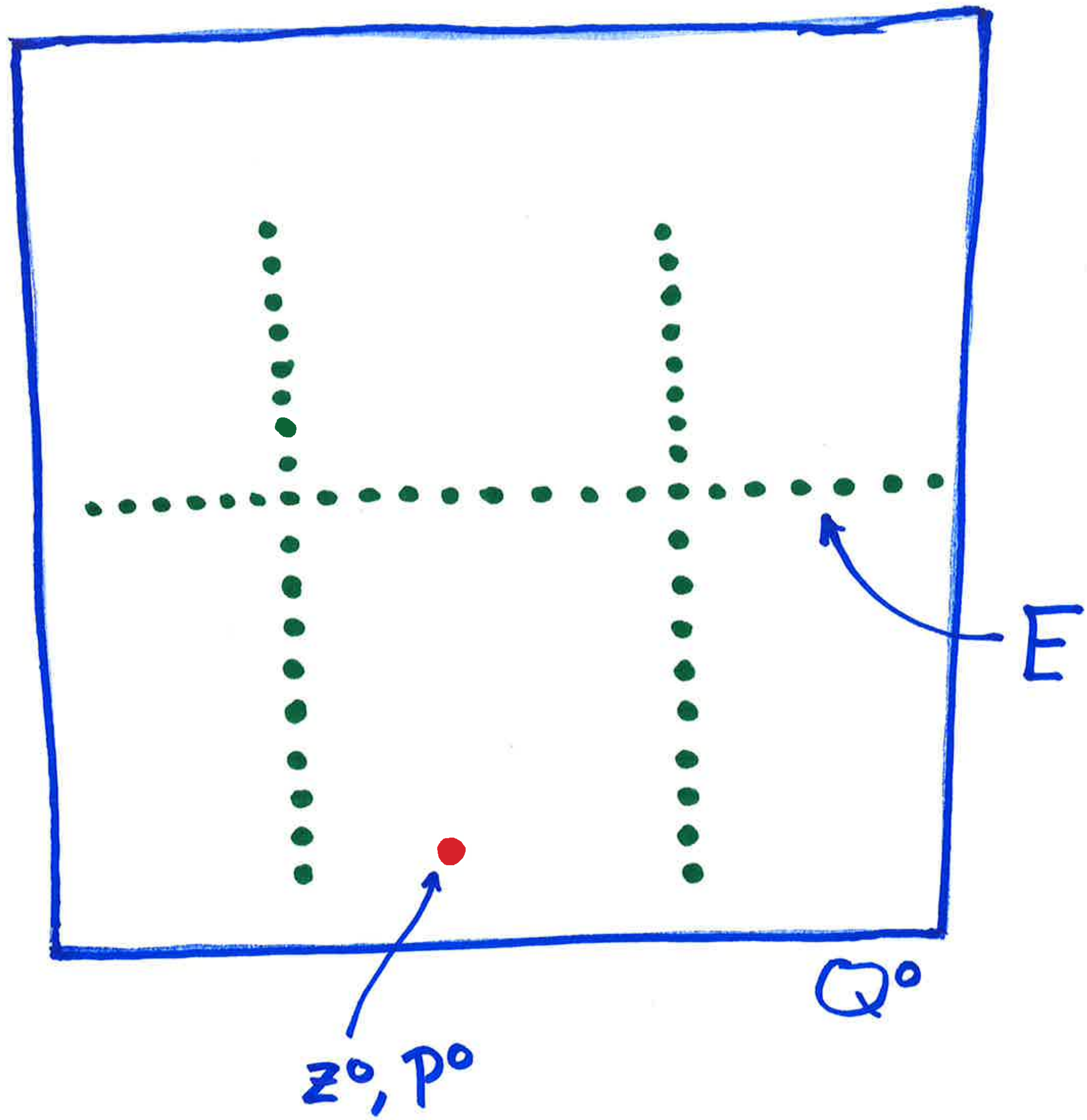
$$F = \sum_{Q} \theta_Q F_Q$$

has ALL THE DESIRED PROPERTIES.

WE'RE DONE WITH EXAMPLE 2.

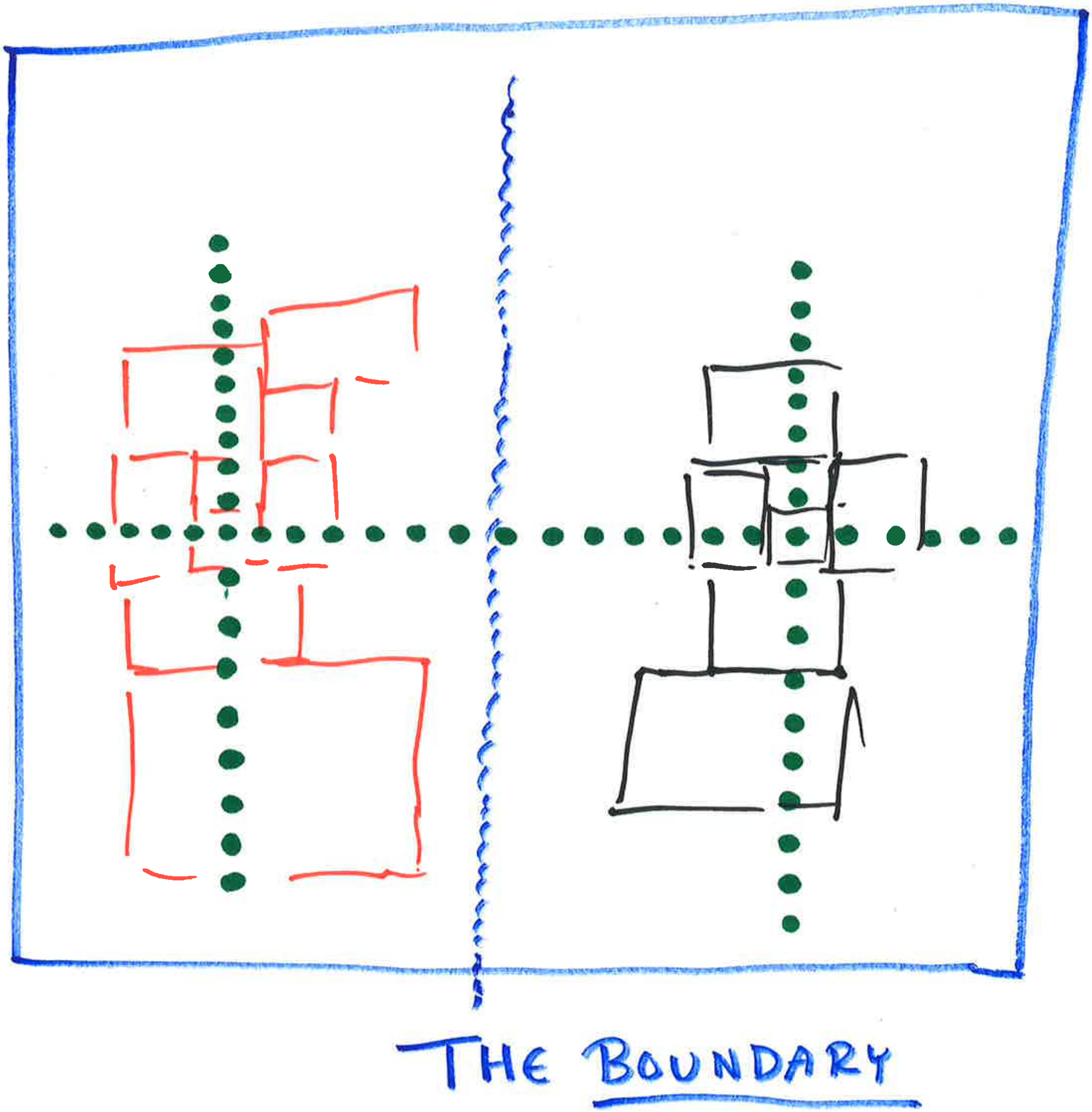
EXAMPLE 3 :

AGAIN, WE WORK IN $L^{2,p}(\mathbb{R}^2)$, $p > 2$.



LABEL $a = \emptyset$ (EMPTY SET)

THE CZ CUBES IN EXAMPLE 3



To produce all the F_Q,

WE TREAT ALL THE

ORANGE CUBES □

AS IN EXAMPLE 2,

& WE TREAT ALL THE

BLACK CUBES □

AS IN EXAMPLE 2.

BY THE WAY,

BLACK AND ORANGE

ARE THE

SCHOOL COLORS OF

PRINCETON

As in Example 2,

INFORMATION FLOWS OUTWARD,

FROM THE SMALLEST

ORANGE CUBE

TO ALL THE ORANGE CUBES,

and

FROM THE SMALLEST

BLACK CUBE

TO ALL THE BLACK CUBES.

WE'RE DONE WITH

EXAMPLE 3.

MORAL OF THE STORY

CZ CUBES $Q^{\#}$ at the SMALLEST LENGTHSCALES CARRY THE CRUCIAL INFORMATION NEEDED TO DEFINE MUTUALLY CONSISTENT POLYS P_Q FOR ALL THE CZ CUBES Q.

WE WILL TREAT EACH SUCH $Q^{\#}$ INDEPENDENTLY, AND FIND POLYS. $P_{Q^{\#}}$.

For each Q at a larger lengthscale, we will find a nearby $Q^{\#}$, and define $P_Q := P_{Q^{\#}}$.

HAVING LEARNED OUR LESSONS

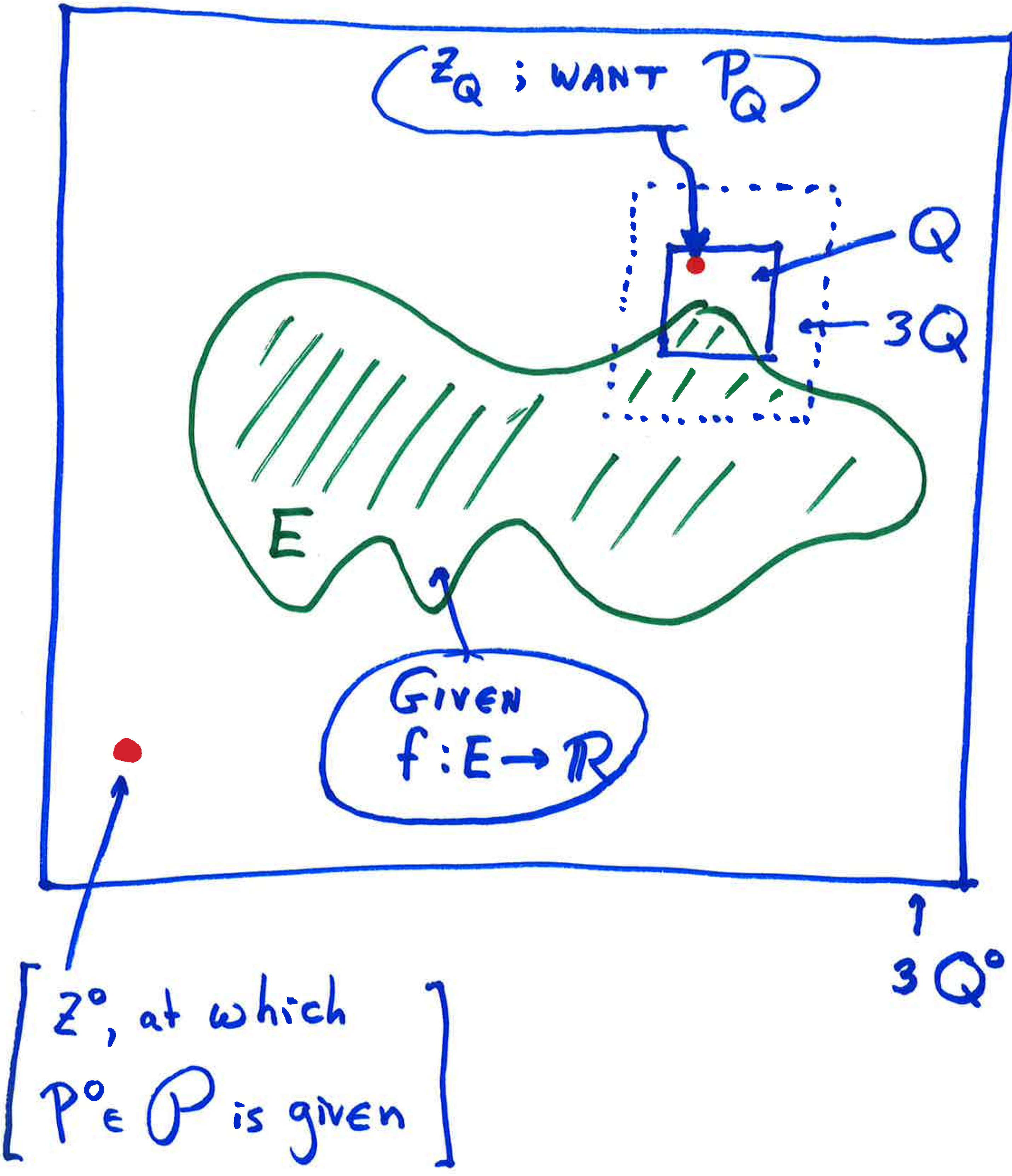
FROM THE ABOVE

3 EXAMPLES

WE NOW RETURN TO THE

GENERAL CASE.

RECALL THE SITUATION:

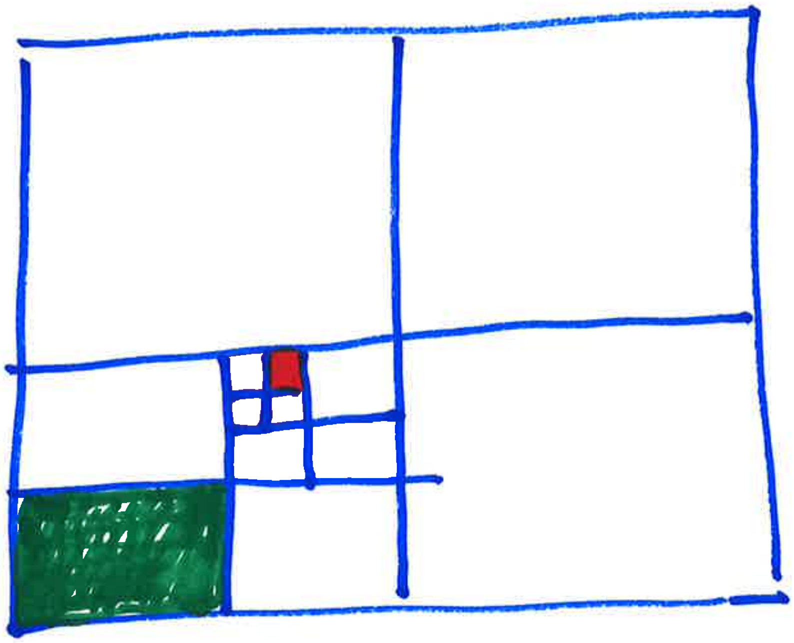


z_Q ; WANT P_Q
Q
3Q
E
GIVEN $f : E \to \mathbb{R}$
$3Q^\circ$
z°, at which $P^\circ \in \mathcal{P}$ is given

LET US PICK OUT THE CZ CUBES at the "SMALLEST LENGTHSCALE".

DEFINITION:

A CZ CUBE $Q^{\#}$ is CALLED A **KEYSTONE CUBE** IF EVERY CZ CUBE Q THAT MEETS $100Q^{\#}$ IS AT LEAST AS BIG AS $Q^{\#}$.



AN EXAMPLE:

The RED SQUARE IS KEYSTONE,

The GREEN SQUARE IS NOT.

PROBLEM:

How to associate a poly $P_Q \in \mathcal{P}$

to each CZ cube Q,

so that

$$\boxed{\text{Each } P_Q \text{ arises from } (f, P^\circ) \text{ by applying a linear map of ABD}}$$

and

$$\boxed{\text{The } P_Q \text{ are mutually consistent.}}$$

?

THE PLAN

STEP 1 : Associate to each CZ cube Q a nearby KEYSTONE CUBE $Q^{\#} = K(Q)$.

<u>STEP 2</u>:

ASSIGN TO EACH KEYSTONE CUBE $Q^{\#}$ a POLY. $P_{Q^{\#}} \in \mathcal{P}$, treating the different Keystone cubes independently of one another.

<u>STEP 3</u>: For each $Q \in CZ$,

define

$$P_Q = P_{Q^\#},$$

where $Q^\# = K(Q)$.

WE MAKE AN EXCEPTION

FOR THE (UNIQUE)

CZ CUBE CONTAINING

THE BASE POINT z^0.

TO THAT CUBE, WE ASSIGN

THE POLY P^0.

LET'S IGNORE THIS EXCEPTION

FROM NOW ON.

CARRYING OUT THE PLAN

Step 1: Finding $Q^{\#} = K(Q)$.

Let Q be a CZ cube.

> If Q is already KEYSTONE, then we set $Q^{\#} = Q$, and we are done

Suppose Q is NOT KEYSTONE.

If Q isn't KEYSTON,

then there's a "PATH"

$$Q = Q_1 \longleftrightarrow Q_2 \longleftrightarrow \cdots \longleftrightarrow Q_L = \tilde{Q},$$

with each Q_i a CZ CUBE,

s.t.

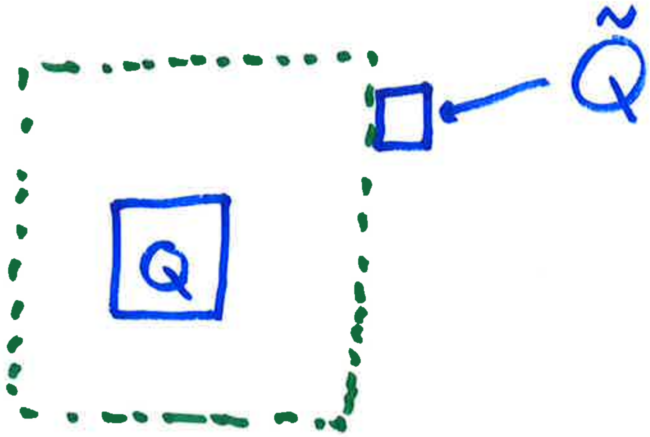
$L \le C$

and

$$\delta_{\tilde{Q}} \le \frac{1}{2}\delta_Q .$$

WE CALL $\tilde{Q}$ A JUNIOR PARTNER of Q.

ONE WAY TO FIND A JUNIOR PARTNER



DILATE Q ABOUT ITS CENTER UNTIL WE ENCOUNTER A CZ CUBE $\tilde{Q}$ SMALLER THAN Q.

WE DILATE BY AT MOST A FACTOR 100.

So EITHER Q IS KEYSTONE,

or it has a JUNIOR PARTNER $\tilde{Q}$.

If there's a junior partner $\tilde{Q}$,

then either $\tilde{Q}$ is KEYSTONE,

or <u>it</u> has a junior partner $\tilde{\tilde{Q}}$.

If we reach $\tilde{\tilde{Q}}$, then either

$\tilde{\tilde{Q}}$ is KEYSTONE, or it has

a junior partner $\tilde{\tilde{\tilde{Q}}}$.

ETC.

CONTINUE IN THIS WAY UNTIL WE REACH A KEYSTONE CUBE.

The process has to stop eventually, because there are only finitely many CZ cubes.

RECALL THAT EACH NON-KEYSTONE CUBE $\tilde{\overset{\cdots}{\tilde{Q}}}$ IS JOINED TO ITS JUNIOR PARTNER $\tilde{\tilde{\overset{\cdots}{\tilde{Q}}}}$ by a "path"

$$\tilde{\overset{\cdots}{\tilde{Q}}} = Q_1 \longleftrightarrow Q_2 \longleftrightarrow \cdots \longleftrightarrow Q_L = \tilde{\tilde{\overset{\cdots}{\tilde{Q}}}}$$

with $L \leq C$.

CONCATENATING ALL THESE PATHS,

we arrive at a path

$$Q = Q_1 \leftrightarrow \cdots \leftrightarrow Q_S = Q^\#,$$

where

$\boxed{Q^\# \text{ is KEYSTONE}}$,

$\boxed{\text{Each } Q_s \text{ is a CZ cube}}$,

and

$$\boxed{\begin{array}{l} \delta_{Q_s} \leq C e^{-c(s-t)} \delta_{Q_t} \\ \text{for } \quad s \geq t. \end{array}}$$

We can take $K(Q) = Q^{\#}$.

Thus, we have associated to each CZ cube Q a nearby keystone cube $K(Q)$.

So we might declare victory over Step 1.

IN FACT, WE HAVE TO BE MUCH MORE CAREFUL.

We want to make sure that there are at most CN BORDER DISPUTES.

A <u>BORDER DISPUTE</u> is a pair of adjacent CZ cubes $Q \leftrightarrow Q'$ such that $K(Q) \neq K(Q')$.

THIS CAN BE DONE.

So we can carry out STEP 1 of the PLAN:

$$\boxed{\text{We have assigned to each CZ cube } Q \text{ a nearby KEYSTONE CUBE } Q^{\#} = K(Q).}$$

STEP 2:

LET $Q^{\#}$ BE A KEYSTONE CUBE.

WE LOOK FOR A POLY. $P_{Q^{\#}} \in \mathcal{P}$ TO ASSIGN TO $Q^{\#}$.

OUR $P_{Q^{\#}}$ IS SUPPOSED TO ARISE FROM $(f, P^{\circ}) \in L^{m,p}(E) \oplus \mathcal{P}$ BY APPLYING A LINEAR MAP OF ABD.

FOR ANY GIVEN $P^{\#} \in \mathcal{P}$,

WE TRY SETTING

$$\boxed{P_{Q^{\#}} = P^{\#}}$$

AND SEE HOW WELL

IT PERFORMS.

ONCE WE KNOW HOW

TO DO THAT,

$$\boxed{\text{WE CAN TAKE } P_Q^{\#} \text{ TO BE THE } P^{\#} \text{ THAT PERFORMS BEST}}$$

(or NEARLY SO).

TO GUESS HOW WELL $P^{\#}$ PERFORMS, WE EXAMINE WHAT HAPPENS ON $100\,Q^{\#}$.

LET'S SEE HOW $100\,Q^{\#}$ LOOKS.

$\boxed{\text{LET } Q_1, \ldots, Q_{\nu_{MAX}} \text{ BE THE CZ CUBES THAT MEET } 100Q^{\#}.}$

BECAUSE $Q^{\#}$ IS KEYSTONE, THERE ARE AT MOST C SUCH CUBES, AND THEY ARE ALL ABOUT THE SAME SIZE.

That is,

$$\boxed{\nu_{MAX} \leq C}$$

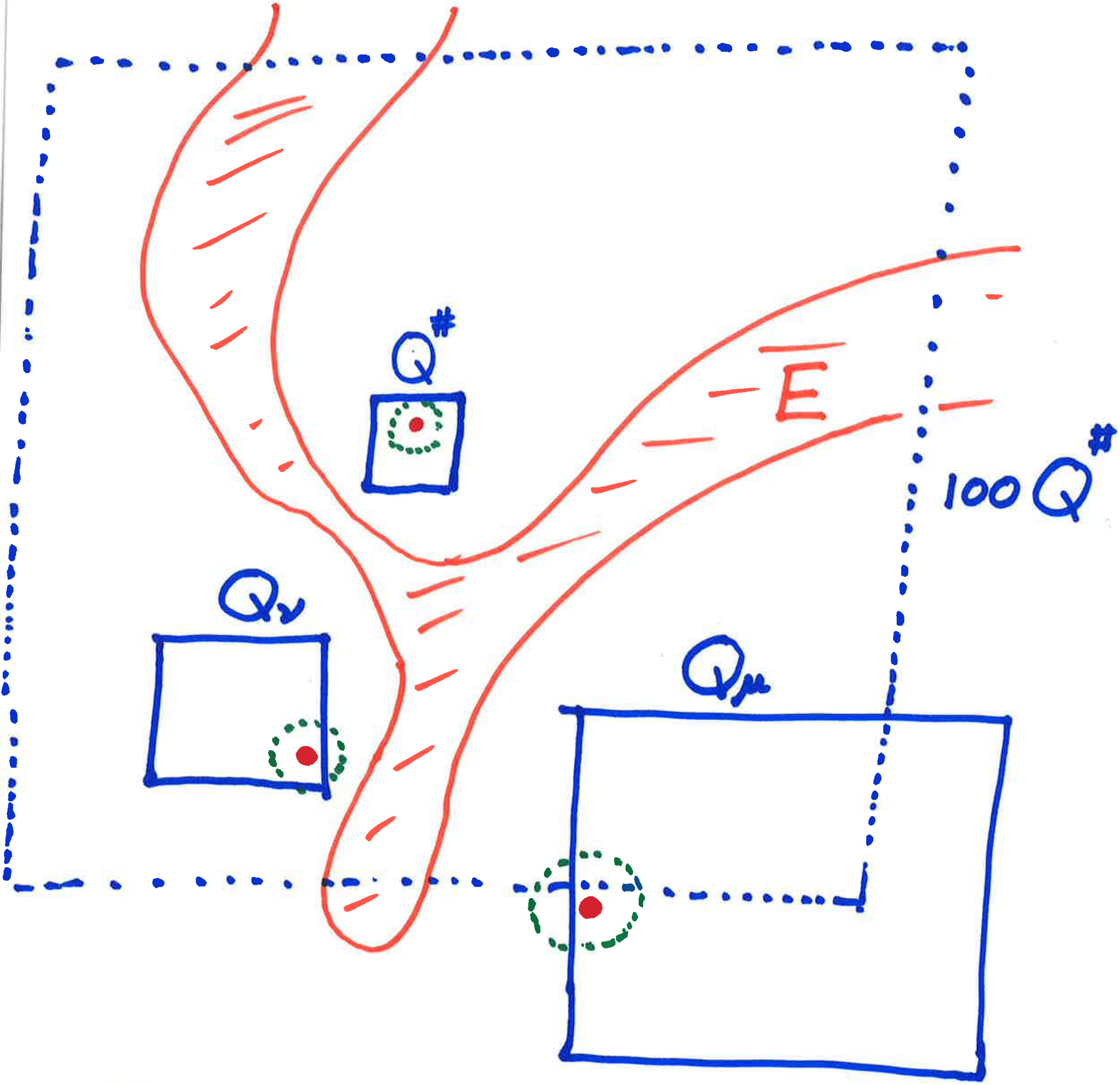
and

$$\boxed{\delta_{Q^\#} \leq \delta_{Q_\nu} \leq C\,\delta_{Q^\#} \quad \text{for } \nu = 1, \dots, \nu_{MAX}.}$$

In each Q_ν, we picked out a base pt. z_{Q_ν}, NOT TOO CLOSE TO E

$$\boxed{\text{DIST.}\left(z_{Q_\nu}, E\right) \geq c\,\delta_{Q_\nu} \geq c\,\delta_{Q^\#}}$$

LET'S WRITE z_ν IN PLACE OF z_{Q_ν}.



EACH BALL

$B(z_\nu, c\delta_{Q^\#})$

AVOIDS E

Now we know how 100 $Q^{\#}$ looks.

Let's guess how well a given "candidate" $P^{\#} \in \mathcal{P}$ performs on $100\,Q^{\#}$ if we decide to set $P_{Q^{\#}} = P^{\#}$.

GIVEN $P^{\#}$, WE LOOK FOR A FUNCTION $F^{\#} \in L^{m,p}(100\,Q^{\#})$

SUCH THAT

$$\boxed{\begin{aligned} &F^{\#} = f \text{ on } E \cap 100\,Q^{\#} \\ &J_{z_{Q^{\#}}}(F) = P^{\#} \end{aligned}}$$

with $L^{m,p}$-NORM (ALMOST) AS SMALL AS POSSIBLE.

OUR CANDIDATE $P^{\#}$

PERFORMS WELL

IF $\| F^{\#} \|_{L^{m,p}(100Q^{\#})}$

IS SMALL.

So, given $P^{\#}$, we must

- Find $F^{\#}$

- Guess $\|F^{\#}\|_{L^{m,p}(100Q^{\#})}$

- Pick $P^{\#}$ to make small

FINDING $F^{\#}$

RECALL, WE CAN SOLVE

$LIP(Q_\nu, E \cap Q_\nu, z_\nu)$

for each ν, thanks to

INDUCTION HYPOTHESIS.

(MAIN LEMMA HOLDS FOR LABELS $\mathcal{A}' < \mathcal{A}$)

Given the function $f|_{E \cap 3Q_\nu}$,

and given a poly.

$P_\nu \in \mathcal{P}$ associated to z_ν,

we can find a nearly optimal

$F_\nu \in L^{m,p}((1.01)Q_\nu)$ s.t.

$$\boxed{\begin{array}{l} F_\nu = f \text{ on } E \cap (1.01)Q_\nu \ \& \\ J_{z_\nu}(F_\nu) = P_\nu . \end{array}}$$

WE THEN DEFINE OUR $F^{\#}$ BY PATCHING TOGETHER THE F_ν, USING A PARTITION OF UNITY ADAPTED TO THE Q_ν:

$$\boxed{F^{\#} = \sum_\nu \theta_\nu F_\nu.}$$

How to Guess the P_ν ?

Since the Geometry of the

Q_ν , z_ν

is so Simple,

we may as well

just set

$P_\nu = P^{\#}$ for each ν.

SO, GIVEN A CANDIDATE $\mathcal{P}^\#$

FOR THE OFFICE OF $\mathcal{P}_{Q^\#}$,

WE CAN NOW PRODUCE

A FUNCTION $F^\# \in L^{m,p}(100Q^\#)$.

WE LIKE OUR $\mathcal{P}^\#$ IF

$\|F^\#\|_{L^{m,p}(100Q^\#)}$ IS SMALL.

HOW TO GUESS $\|F^{\#}\|_{L^{m,p}(100Q^{\#})}$?

USE THE PATCHING LEMMA:

$$\|F^{\#}\|^{p}_{L^{m,p}(100Q^{\#})} \leq$$

$$C \sum_{\nu} \|F_{\nu}\|^{p}_{L^{m,p}((1.01)Q_{\nu})}$$

$$+ C \sum_{\mu \leftrightarrow \nu} \delta_{\mu}^{-mp} \|P_{\mu} - P_{\nu}\|^{p}_{L^{p}(Q_{\mu})}$$

WHERE . . .

- $P_\mu = J_{z_\mu}(F_\mu)$

- $P_\nu = J_{z_\nu}(F_\nu)$

- $\mu \longleftrightarrow \nu$ MEANS Q_μ & Q_ν TOUCH.

WE HAVE SET

$P_\mu = P_\nu = P^\#$,

SO THE $\| P_\mu - P_\nu \|$ TERM

IN THE PATCHING LEMMA

IS $= 0$

So the PATCHING LEMMA gives:

$$\boxed{\|F^{\#}\|^{p}_{L^{m,p}(100Q^{\#})} \le C \sum_{\nu} \|F_{\nu}\|^{p}_{L^{m,p}((1.01)Q_{\nu})}.}$$

WE GUESS THAT LHS, RHS HAVE SAME ORDER OF MAG.

So if we can compute

the order of magnitude

of $\|F_\nu\|^p_{L^{m,p}((1.01)Q_\nu)}$

for each ν,

then we can <u>guess</u>

the order of magnitude

of $\|F^\#\|^p_{L^{m,p}(100Q^\#)}$.

RECALL :

For each Q_ν, we've solved

$LIP(Q_\nu, E \cap 3Q_\nu, z_\nu)$

by INDUCTIVE HYP.

(MAIN LEMMA HOLDS
FOR LABELS $\mathcal{A}' < \mathcal{A}$)

In particular,

INDUCTIVE HYP.

gives a

FORMULA

FOR THE

NORM !

That formula tells us that our F_ν, defined as an almost optimal solution of

$$\boxed{\begin{aligned} & F_\nu = f \text{ on } E \cap (1.01)Q_\nu \\ & J_{z_\nu}(F_\nu) = P^\#, \end{aligned}}$$

has $L^{m,p}$ - NORM GIVEN BY ...

$$\boxed{\|F_\nu\|^p_{L^{m,p}((1.01)Q_\nu)} \sim \sum_{\ell=1}^{L(\nu)} |\xi_{\ell,\nu}(f, P^\#)|^p}$$

for known linear functionals

$\xi_{1,\nu}, \ldots, \xi_{L(\nu),\nu}$.

Substituting this

formula for $\| F_\nu \|_{L^{m,p}((1.01)Q_\nu)}$

into our (CONJECTURED)

formula for $\| F^\# \|_{L^{m,p}(100Q^\#)}$,

WE ARRIVE AT THE

GUESS :

$$\|F^\#\|^p_{L^{m,p}(100Q^\#)} \sim \sum_{\nu} \sum_{\ell} |\xi_{\ell,\nu}(f, P^\#)|^p$$

FORMULA

So at last, we know how to guess whether our candidate $P^{\#}$ performs well:

$P^{\#}$ performs well

$\Updownarrow$

RHS of formula ☺ is small.

Now that we know how to

Evaluate Candidates

$P^\# \in \mathcal{P}$,

WE CAN PICK THE WINNER,

AND DECLARE IT

TO BE $P_{Q^\#}$.

LET'S SEE HOW THAT GOES...

$$\boxed{\begin{array}{l}\text{WE ARE GIVEN} \\ (f, P^0) \in L^{m,p}(E) \oplus \mathcal{P}.\end{array}}$$

FOR A GIVEN KEYSTONE CUBE $Q^\#$,

WE WANT TO PICK $P_{Q^\#}$

FROM AMONG THE

CANDIDATES $P^\# \in \mathcal{P}$.

As in the C^m case,

we restrict attention

to candidates $P^\# \in \mathcal{P}$ s.t.

$$\partial^\alpha (P^\# - P^0) \equiv 0 \text{ for } \alpha \in \mathcal{A}.$$

(Recall: Our LIP (Q^0, E, z^0) carries label $\mathcal{A}$.)

We call such $P^\#$

coherent with P^0.

AMONG ALL $P^{\#} \in \mathcal{P}$

THAT ARE COHERENT W/ P^{0},

WE PICK ONE THAT MAKES

RHS OF ☺

$$\equiv \sum_{\nu} \sum_{\ell} \left| \xi_{\ell,\nu}(f, P^{\#}) \right|^{p}$$

AS SMALL AS POSSIBLE

(up to a factor C).

THAT WILL BE OUR $P_{Q^{\#}}$.

CRUCIAL POINT:

We can pick our

WINNING $P^\#$

TO DEPEND

LINEARLY

on (f, P°),

thanks to the following

simple result:

Let $\xi_1, \dots, \xi_K$ be linear functionals on $\mathbb{R}^N \oplus \mathbb{R}^D$.

Given $v \in \mathbb{R}^N$, we want to find $w \in \mathbb{R}^D$ that (almost) minimizes

$$\sum_{k=1}^{K} |\xi_k(v, w)|^p.$$

LEMMA:

There exists a linear map

$T : \mathbb{R}^N \to \mathbb{R}^D$

such that

$$\sum_k |\xi_k(v, Tv)|^p \leq C \sum_k |\xi_k(v, w)|^p$$

for all $v \in \mathbb{R}^N$, $w \in \mathbb{R}^D$.

Here, C depends only on D.

Thus, the vector

$$W^* = T v$$

MINIMIZES

$$\sum_{1}^{K} \left| \xi_k (v, w) \right|^p$$

up to a factor C,

& W^* depends linearly on v.

To apply the LEMMA,

we think of

(f, P^0) as a vector in $\mathbb{R}^N$,

and we think of

$$\left(\partial^\alpha P^\#(x_{Q^\#})\right)_{\alpha \notin \mathcal{A}}$$

as a vector in $\mathbb{R}^D$.

The LEMMA then shows that

AN ESSENTIALLY OPTIMAL $P^{\#}$ (COHERENT WITH P°) CAN BE TAKEN TO DEPEND LINEARLY on (f, P°).

PROOF OF THE LEMMA:

INDUCTION ON D.

$$\boxed{\text{The case } D=1 \text{ follows easily from Hölder's inequality.}}$$

$$\boxed{\text{The induction step } D \rightsquigarrow D+1 \text{ follows easily from the case } D=1.}$$

LET'S BELIEVE IT.

$$\boxed{\text{So we have (finally!) picked our } P_{Q^\#}.}$$

As required, it depends LINEARLY on (f, P°).

However, it's probably NOT ABD.

SOLUTION:

ADD NEW ASSISTS!

For each KEYSTONE $Q^{\#}$

and each α $(|\alpha| \leq m-1)$,

take the functional

$$(f, P^{\circ}) \Big|_{P^{\circ}=0} \longmapsto \partial^{\alpha} P_{Q^{\#}}(z_Q)$$

to be a NEW ASSIST.

Trivially,

$$(f, P^\circ) \longmapsto P_{Q^\#}$$

is then ABD

with respect to

the NEW ASSISTS.

OF COURSE WE MUST PROVE that our ENLARGED SET of assists Ω SATISFIES the DEFINING CONDITION

$$\boxed{\sum_{\omega \in \Omega} dp(\omega) \leq CN}$$

where $N = \#(E)$

and $dp(\omega) = \#$ of NONZERO COEFF's in the formula for ω.

LET'S BELIEVE IT!

So NOW WE HAVE Carried out

STEP 2

by assigning a poly $P_{Q^\#}$

to each KEYSTONE CUBE $Q^\#$.

The $P_{Q^\#}$ depend on (f, P^o).

The maps $(f, P^o) \mapsto P_{Q^\#}$

are linear ABD

(with respect to the enlarged set of assists).

On to Step 3!

STEP 3 IS EASY!

Given a CZ cube Q, we have produced a KEYSTONE CUBE $Q^{\#} = K(Q)$ in STEP 1.

Given a KEYSTONE CUBE $Q^{\#}$, we have produced a poly. $P_{Q^{\#}} \in \mathcal{P}$ in STEP 2.

STEP 3 CONSISTS SIMPLY OF DEFINING

$$P_Q := P_{Q^\#} \quad (Q^\# = K(Q))$$

for each CZ CUBE Q.

So WE ARE DONE WITH STEP 3!

BY CARRYING OUT

STEPS 1, 2, 3,

WE HAVE NOW ASSIGNED

A POLY. $P_Q \in \mathcal{P}$

TO EACH CZ CUBE Q.

WHERE DO WE STAND?

We've fixed $\mathcal{a} \neq \mathcal{M}$ monotonic,

and we've assumed the

INDUCTION HYPOTHESIS:

The MAIN LEMMA holds

for all labels $\mathcal{a}' < \mathcal{a}$.

OUR TASK IS TO PROVE THE MAIN LEMMA for $\mathcal{A}$.

That is, WE MUST SOLVE the LOCAL INTERPOLATION PROB.

$LIP(Q^\circ, E, z^\circ)$

whenever that problem carries the label $\mathcal{A}$.

Fix such a $\mathcal{LIP}(Q^\circ, E, z^\circ)$.

Given $(f, P^\circ) \in L^{m,p}(E) \oplus \mathcal{P}$,

our task is to produce a fn.

$F \in L^{m,p}((1.01)Q^\circ)$ such that

$$\boxed{\begin{array}{l} F = f \text{ on } E \cap (1.01)Q^\circ \\ \quad \& \\ J_{z_0}(F) = P^\circ. \end{array}}$$

Our F must have norm

$$\|F\|_{L^{m,p}((1.01)Q^\circ)}$$

as small as possible,
up to a factor C.

Also, our F must arise

by applying to (f, P°)

a LINEAR MAP OF

ASSISTED BDD DEPTH.

FINALLY, WE MUST PRODUCE A FORMULA for $\|F\|$ OF THE FORM

$$\boxed{\|F\|^{p}_{L^{m,p}((1.01)Q^{\circ})} \sim \sum_{\nu=1}^{\nu_{MAX}} |\xi_{\nu}(f, P^{\circ})|^{p}}$$

where ...

$\boxed{\xi_1, \dots, \xi_{\nu_{MAX}} \text{ ARE LINEAR FUNCTIONALS OF ASSISTED BDD. DEPTH}}$

and

$$\boxed{\nu_{MAX} \leq C \cdot \#(E).}$$

THAT'S WHAT WE

HAVE TO DO.

WHAT HAVE WE DONE?

We've made a Calderón-Zygmund

decomposition of $3Q^\circ$.

In each CZ cube Q,

we've fixed a base pt. z_Q.

By carrying out

STEPS 1, 2, 3,

WE'VE ASSIGNED A POLY.

$P_Q \in \mathcal{P}$

TO EACH CZ CUBE Q.

OUR CZ STOPPING RULE GUARANTEES THAT EACH OF THE LOCAL PROBLEMS $LIP(3Q, E \cap 3Q, z_Q)$ CAN BE SOLVED $(Q \in CZ)$.

Therefore, from $(f|_{E\cap 3Q}, P_Q)$,

we produce a

LOCAL INTERPOLANT F_Q

such that

$$\boxed{\begin{array}{l} F_Q = f \text{ on } E \cap (1.01)Q \\ \text{and} \\ J_{z_Q}(F_Q) = P_Q . \end{array}}$$

Our F_Q has NORM

$$\|F_Q\|_{L^{m,p}((1.01)Q)}$$

as SMALL AS POSSIBLE

up to a factor C,

and the map

$$(f|_{E\cap 3Q}, P_Q) \mapsto F_Q$$

is LINEAR ABD.

Moreover, for each CZ cube Q, we have a

FORMULA FOR THE NORM:

$$\boxed{\| F_Q \|^p_{L^{m,p}((1.01)Q)} \sim \sum_{\nu=1}^{\nu_{\max}(Q)} \left| \xi^Q_\nu (f, P_Q) \right|^p .}$$

USING A WHITNEY PARTITION OF UNITY associated to our CZ cubes Q, WE Combine our local interpolants F_Q into

$$\boxed{F := \sum_Q \theta_Q F_Q.}$$

This F will satisfy

$$\boxed{\begin{gathered} F = f \text{ on } E \cap (1.01)Q^\circ \\ \text{and} \\ J_{z^\circ}(F) = P^\circ. \end{gathered}}$$

MOREOVER,

$(f, P^\circ) \mapsto F$

is LINEAR ABD.

WHAT MORE

MUST WE DO?

WE MUST PRODUCE A FORMULA FOR THE NORM:

$$\boxed{\|F\|^p_{L^{m,p}((1.01)Q^\circ)} \sim \sum_\nu |\xi_\nu(f, P^\circ)|^p}$$

for suitable ABD functionals

$\xi_1, \ldots, \xi_{\nu_{MAX}}$.

FOR ANY "COMPETITOR"

$\tilde{F} \in L^{m,p}((1.01)Q^\circ)$ such that

$$\boxed{\begin{array}{l} \tilde{F} = f \text{ on } E \cap (1.01)Q^\circ \\ \quad \& \\ J_{z^\circ}(\tilde{F}) = P^\circ \end{array}}$$

WE MUST SHOW THAT

$$\|F\|_{L^{m,p}} \leq C \|\tilde{F}\|_{L^{m,p}}.$$

Once we do all that,

We are done!

OUTLINE OF THE REST OF THE PROOF:

---

(1) PRODUCE A CONJECTURED FORMULA FOR THE NORM, AND PROVE THAT

$$\|F\|_{L^{m,p}}^{p} \leq C \sum_{\nu} |\xi_{\nu}(f, P^{0})|^{p}$$

(NOT HARD)

② PROVE THAT ANY COMPETITOR $\tilde{F}$ SATISFIES

$$\sum_{\nu} |\xi_\nu(f, P^\circ)|^p \leq C \|\tilde{F}\|^p_{L^{m,p}}.$$

(A LONG STORY, MAKING USE OF OUR WISE CHOICE OF THE $\mathcal{P}_Q$ IN STEPS 1, 2, 3)

LET'S JUST BELIEVE IT !

Taking $\tilde{F} = F$ in (2),

WE SEE THAT

$$\|F\|^p_{L^{m,p}} \geq c \sum_\nu |\xi_\nu(f, P^\circ)|^p.$$

Together with (1), ~~we~~ this yields our FORMULA for the NORM:

$$\boxed{\|F\|^p_{L^{m,p}} \sim \sum_\nu |\xi_\nu(f, P^\circ)|^p.}$$

Thanks to our

FORMULA for the NORM,

$$\textcircled{2} \implies \boxed{\|F\|_{L^{m,p}} \le C \|\tilde{F}\|_{L^{m,p}}}$$

for ANY COMPETITOR $\tilde{F}$.

SO WE ARE DONE!

(Except that we never discussed HOW TO PROVE ②).

THIS (FINALLY!)

CONCLUDES OUR DISCUSSION

OF THE PROOF OF

TODAY'S THM.

P.S. RECALL THAT OUR PROOF USES THE

TRUE σ.

THERE ARE NO σ_ℓ OR T_ℓ AS IN THE C^m CASE.

THEREFORE,

TO COMPUTE ANYTHING,

WE NEED

SUBSTANTIAL
NEW IDEAS.

AGAIN,

LET'S JUST

DECLARE

VICTORY.

SO WE'VE FINISHED WITH SOBOLEV SPACES

[FOR TODAY — BUT

WHAT ABOUT

$L^{m,p}(\mathbb{R}^n)$, $p \le n$?].

THANK

YOU !